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Dynamic Programming Principle for Optimal Control of Uncertain Random Differential Equations and its Application to Optimal Portfolio Selection

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ABSTRACT

This study **aimed** to examine an uncertain stochastic optimal control problem premised on an uncertain stochastic process. The proposed approach is used to solve an optimal portfolio selection problem. This paper's research is **relevant** because it outlines the procedure for solving optimal control problems in uncertain random environments. We implement Bellman's principle of optimality **method** in dynamic programming to derive the principle of optimality. Then the resulting Hamilton-Jacobi-Bellman equation (the equation of optimality in uncertain stochastic optimal control) is used to solve a proposed portfolio selection problem. The **results** of this study show that the dynamic programming principle for optimal control of uncertain stochastic differential equations can be applied in optimal portfolio selection. Also, the study results indicate that the optimal fraction of investment is independent of wealth. The main **conclusion** of this study is that, in Itô-Liu financial markets, the dynamic programming principle for optimal control of uncertain stochastic differential equations can be applied in solving the optimal portfolio selection problem.
Keywords: randomness; uncertainty; uncertain random differential equations; dynamic programming; optimal control; portfolio selection; equation of optimality; Itô-Liu financial markets

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ОРИГИНАЛЬНАЯ СТАТЬЯ

Принцип динамического программирования для оптимального управления неопределенными случайными дифференциальными уравнениями и его применение к оптимальному выбору портфеля

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АННОТАЦИЯ

Целью данного исследования было изучение поведения финансовых рынков как неопределенной стохастической задачи оптимального управления, основанной на неопределенном стохастическом процессе. Предлагаемый подход используется для решения задачи оптимального выбора портфеля. Исследование, проведенное в данной работе, является актуальным, поскольку оно описывает процедуру решения задач оптимального управления в неопределенных случайных средах. Авторы реализуют **метод** принципа оптимальности Беллмана в динамическом программировании для вывода принципа оптимальности. Затем полученное уравнение Гамильтона–Якоби–Беллмана (уравнение оптимальности в неопределенном стохастическом оптимальном управлении) используется для решения предложенной задачи выбора портфеля. **Результаты** данного исследования показывают, что принцип динамического программирования для оптимального управления неопределенными стохастическими дифференциальными уравнениями может быть применен при оптимальном выборе портфеля. Кроме того, результаты исследования указывают на то, что оптимальная доля инвестиций не зависит от состояния. Основной **вывод** данного исследования заключается в том, что на финансовых рынках Ито-Лю принцип динамического программирования для оптимального управления неопределенными стохастическими дифференциальными уравнениями может быть применен при решении задачи оптимального выбора портфеля.

Ключевые слова: случайность; неопределенность; неопределенные случайные дифференциальные уравнения; динамическое программирование; оптимальное управление; выбор портфеля; уравнение оптимальности; финансовые рынки Ито-Лю

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Introduction

Generic uncertainty experienced in natural and physical processes manifests in different forms [1]. In this modern world, the primary forms of generic uncertainty are fuzziness, randomness, uncertainty, and the interaction effects between any of these three [1, 2]. Randomness describes the state of a system that is entirely unknown due to a lack of information [3]. Similarly, [4] propounded that randomness is an objective indeterminacy. Randomness is modelled by probability theory, which is defined as a branch of pure mathematics that deals with dynamic random phenomena [5, 6]. Conceptually, probability theory is implemented when the sample size is big enough to approximate the probability distribution from existing frequency [6]. Probability theory is the bedrock of stochastic finance theory. Fuzziness is the vagueness surrounding the description of the meaning of events, phenomena, and statements themselves [3]. Fuzzy set theory models fuzziness [7]. The application of fuzzy theory in finance theory led to the emergence of fuzzy finance theory.

In practice, some knowledge or information is typically shown by human semantic terms like ‘‘stock price is about $ 28’’ [2, 5, 8]. About $ 28 might mean any number close to $ 28, which is imprecise. Existing literature has indicated that these ‘‘unknown constants’’ and ‘‘unsharp concepts’’ behave neither like fuzziness nor randomness [2]. This phenomenon is called uncertainty or Liu’s uncertainty [5, 8], and [4] postulated that uncertainty is subjective indeterminacy. The lack of precise or sufficient knowledge about realities identifies uncertainty. It should be noted that uncertainty is different from randomness and fuzziness.

To model uncertainty, [8] introduced uncertainty theory. Uncertainty theory is an axiomatic branch of mathematics that analyzes uncertain phenomena. Uncertainty theory is applied when the sample size is missing or too small to approximate the probability distribution. Domain specialists are asked to examine their belief degrees of every event happening. People typically overvalue odd events. Hence, belief degrees can have a greater variance than the real frequency. In this instance, implementing probability theory results in counter-intuitive results. Uncertainty theory is the foundation of uncertain finance theory. For more information concerning Liu’s uncertainty, the reader is referred to other authors [5, 8].

In probability theory, stochastic processes (e.g., a Brownian motion introduced by Robert Brown in 1827) were designed to analyze the random phenomena dynamics that change with time. Numerous differential equations are powered by a Brownian motion in probability theory. These differential equations are called stochastic differential equations. In uncertainty theory, uncertain processes (e.g., a canonical Liu process [9]) were introduced to analyze the uncertain phenomena and dynamics that change with time. Numerous differential equations are powered by a canonical Liu process in uncertainty theory. These differential equations are called uncertain differential equations.

Randomness and uncertainty often appear simultaneously in a dynamic system [4]. This indicates that more than probability theory or uncertainty theory is needed to deal with systems that exhibit randomness and uncertainty [4, 6]. Chance theory [10] was introduced to deal with sophisticated systems exhibiting uncertainty and randomness. Chance theory is a mathematical methodology comprising uncertainty theory and probability theory [4]. It is defined as a branch of pure mathematics concerned with the analysis of uncertain random phenomena [4, 6].

To analyze the uncertain random phenomena dynamics that change with time in chance theory, [11] introduced an uncertain random process. A myriad of differential equations are powered by an uncertain random process in chance theory. These differential equations are known as uncertain random differential equations and are powered by both a canonical Liu process and a Brownian motion. Uncertain random differential equations describe complex mathematical systems that exhibit randomness and uncertainty [2]. When randomness and uncertainty concurrently appear in dynamical systems, chance theory is an efficient framework to deal with such scenarios.

Since the 1950s, the optimal control theory has been a vital division of modern control theory [4]. These authors [4] further articulated that analyzing optimal control problems is a topic of interest to many researchers, and the analysis has practical connotations. Optimal control problems are usually categorized into two, i.e., optimal control problems associated with adequate information and optimal control problems associated with inadequate information [4]. The parameters of the systems are known, and the dynamics of the systems are described by deterministic differential equations when considering the optimal control problems with complete information [4]. On the other hand, [4] postulated that systems' outcomes or conditions could not be precisely described due to numerous indeterminate factors in the systems' dynamics when considering optimal control problems with inadequate information [4].

Optimal control is one of the areas in mathematics where generic uncertainty issues must be handled cautiously. Applying probability theory in optimal control theory gave birth to stochastic optimal control theory. On the other hand, the application of uncertainty theory in optimal control theory led to the emergence of uncertain optimal control theory. Uncertain optimal control theory and stochastic optimal control theory can be used to address optimal control problems with inadequate information. Basically, uncertain optimal control theory and stochastic optimal control theory are used to solve control problems with subjective indeterminacy and objective indeterminacy, respectively. Stochastic differential equations are crucial in stochastic optimal control theory [12–16]. The application of stochastic optimal control in finance was initiated by [17]. For more information concerning stochastic optimal control, the reader is referred to the works of, but not limited to, [17–23]. Uncertain differential equations are vital in uncertain optimal control theory [24–28]. For more expositions on uncertain optimal control, the reader is referred to, among other sources, [29, 30].

When randomness and uncertainty concurrently appear in dynamic systems, chance theory is an efficient framework to deal with optimal control problems. In the existing literature, there are limited studies that examine uncertain random optimal control problems under the chance theory framework [4, 31]. Premised on chance theory, [32] presented the optimal control model for a multistage uncertain random system. As alluded to earlier, uncertain stochastic differential equations play a critical role in chance theory and, interestingly, in uncertain random markets. [30, 33, 34] are some authors who have examined the dynamic principle for optimal control of uncertain stochastic differential equations in uncertain random markets.

This study examines an uncertain stochastic optimal control problem premised on the notion of uncertain stochastic process. We implement the Bellman's principle of optimality in dynamic programming to derive the principle of optimality, and then the resulting Hamilton-Jacobi-Bellman equation (the equation of optimality in uncertain stochastic optimal control) is used to solve a proposed portfolio selection problem. Previously, [17] examined a portfolio selection problem using stochastic differential equations, while [24] addressed a portfolio selection problem by applying uncertain differential equations. Therefore, this proposed method is a new paradigm for solving optimal control problems in Itô-Liu financial markets.

The results of this study show that the dynamic programming principle for optimal control of uncertain stochastic differential equations can be applied in optimal portfolio selection. Also, the study results indicate that the optimal fraction of investment is independent of wealth. The results are valuable for solving the optimal portfolio selection problem in Itô-Liu financial markets. The main conclusion of this study is that, in Itô-Liu financial markets, the dynamic programming principle for optimal control of uncertain stochastic differential equations can be applied in solving the optimal portfolio selection problem.

The entire paper is organised in the following order. Section 2 contains the preliminaries. In Section 3, an uncertain random optimal control problem is proposed, and the equation of optimality and the principle of optimality are derived and proven. In Section 4, an examination of the portfolio selection problem is done using the dynamic programming approach. Finally, conclusions are articulated in Section 5.

Preliminaries

Some important definitions relating to the concept of uncertain random processes are presented in this section. Let $(\Gamma, \mathcal{L}, M)$ be the universal set on an uncertain space, a σ-algebra, and an uncertain measure, respectively. Also, let $(\Omega, \mathcal{F}, P)$ be the universal set on a probability space, a σ-algebra, and a probability measure, respectively. The chance space is given by $(\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{F}, P)$.

The normality, duality, and monotonicity properties of a chance space were verified by [10].

Definition 1 [10] Given a chance space $(\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{F}, P)$, if $\theta \in \mathcal{L} \otimes \mathcal{F}$ is an event, then

$$Ch(\theta) = \int_0^1 P\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \int \theta\} \geq x\} dx,$$

where $(\Omega, \mathcal{F}, P) and (\Gamma, \mathcal{L}, M)$ are a probability space and an uncertainty space in that order.

Definition 2 [10] An uncertain random variable refers to a function ξ from a chance space $(\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{F}, P)$ to the set $\mathcal{R}$ such that $\varepsilon \in B$ is an event in $\mathcal{L} \otimes \mathcal{F}$ for every Borel set B of real numbers.

Definition 3 [10] For a measurable function ψ, if $F(x)$ is the cumulative distribution function of a random variable κ, τ denotes an uncertain variable and $\psi(x, \tau)$ has an uncertainty distribution $\Psi(x, y)$, the chance distribution of $\psi(\kappa, \tau)$ is given by

$$\Phi(y) = \int_{-\infty}^{\infty} \Psi(x, y) dF(x),$$

where x and y are the realisations of κ and τ, respectively.

Definition 4 [11] Let $(\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{F}, P)$ be a chance space, and let T refer to a totally ordered set. An uncertain random process refers to a function $X_t(\gamma, \omega)$ from $T \times (\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{F}, P)$ to the set of real numbers such that $\{X_t \in B\}$ is an event in $\mathcal{L} \otimes \mathcal{F}$ for every Borel set B of real numbers at each time t.

Definition 5 [11] Assume X_t is an uncertain random process on a chance space $(\Gamma, \mathcal{L}, M) \times (\Omega, \mathcal{F}, P)$. Then, for each fixed $\gamma^* \in \Gamma$ and $\omega^* \in \Omega$, the function $X_t(\gamma^*, \omega^*)$ is called a sample path of the uncertain random process of X_t.

Definition 6 [11] An uncertain random process G_t is a stationary increment uncertain random process if, for $t > 0$, the increment $G_{t+s} - G_s$ are identically distributed uncertain random variables for $s > 0$.

Definition 7 [6] Let C_t and B_t be a one-dimensional canonical process and one-dimensional Brownian motion, respectively, and let Y_t be an uncertain random process. For given functions f, h and g, the differential equation

$$dY_t = f(t, Y_t)dt + g(t, Y_t)dC_t + h(t, Y_t)dB_t,$$

is called an uncertain stochastic differential equation.

Definition 8 [2] Let $X_t = (Z_t, Y_t)^T$ be an uncertain random process. For any partition $P = a = t_1, t_2, t_3 \ldots, t_{k+1} = b$ of the closed interval $[a, b]$, with $a = t_1, t_2, t_3 \ldots, t_{k+1} = b$, the mesh is written as $\Delta = \max_{1<i<k} |t_{i+1} - t_i|$. The Itô-Liu integral of X_t regarding $G_t = (B_t, C_t)$ is defined as follows

$$\int_a^b X_S \mathrm{d}G_S = \lim_{\Delta \to 0} \sum_{i=1}^{N} Z_{t_i} \left(B_{t_{i+1}} - B_{t_i}\right) + \lim_{\Delta \to 0} \sum_{i=1}^{N} Y_{t_i} \left(C_{t_{i+1}} - C_{t_i}\right). \tag{1}$$

given that it exists in mean square and is an uncertain random variable where C_t and B_t are a one-dimensional canonical process and a one-dimensional Wiener process, respectively. In this case, X_t is called Itô-Liu integrable. In particular, when $Y_t \equiv 0$, X_t is called Liu integrable.

Uncertain random optimal control

The problem of optimal control in uncertain random environments concerns choosing a decision that optimizes an objective function associated with an uncertain stochastic process. In the problem, the state variable evolves as an uncertain stochastic differential equation. The concept of an uncertain stochastic integral plays a pivotal role in solving uncertain stochastic differential equations.

Suppose that at any given time s, an uncertain random process $X_s \in \mathcal{R}^k$ defined on a chance space $(\Gamma,\mathcal{L},M)\times(\Omega,\mathcal{F},P)$ can be influenced by a choice of a parameter ζ, which is the decision variable also referred to as the control. The control variable ζ represents the function $\zeta(s,X_s)$ of time s and state X_s.

Let the performance function be

$$J(t,x,\zeta) \equiv E\left[\int_t^T f(s,X_s,\zeta)ds + Q(X_T,T)\right], \tag{2}$$

which represents the anticipated expected optimal reward $J(t,x,\zeta)$ available in $[t,T]$ given that X_s is the state variable. The function $f(s,X_s,\zeta)$ represents the objective function and $Q(X_T,T)$ represents the terminal utility function. The state variable can be expressed as

$$X_s = \int_0^s e(t,X_t,\zeta)dt + \int_0^s \sigma_2(t,X_t,\zeta)dC_t + \int_0^s \sigma_1(t,X_t,\zeta)dB_t.$$

Given $(t,x) \in [0,T]\times \mathcal{R}$, the state equation for $s\in[t,T]$ is given by

$$dX_s = e(s,X_s,\zeta)ds + \sigma_2(s,X_s,\zeta)dC_s + \sigma_1(s,X_s,\zeta)dB_s,\ X_t = x. \tag{3}$$

where $e,\sigma_1\ and\,\sigma_2$ are functions of X_s,ζ and time *s*. Equation (2) is assumed to have a unique solution X_t^*. The value function is given by

$$V(t,x) \equiv \sup_{\zeta} J(t,x,\zeta). \tag{4}$$

Using Equations (1), (2) and (3) above, an uncertain random optimal control problem can be expressed as

$$\begin{cases} V(t,x) \equiv \sup\limits_{\zeta} E\left[\int_t^T f(s,X_s,\zeta)ds + Q(X_T,T)\right] \\ \text{subject to} \\ dX_s = e(s,X_s,\zeta)ds + \sigma_2(s,X_s,\zeta)dC_s + \sigma_1(s,X_s,\zeta)dB_s, \\ X_t = x. \end{cases} \tag{5}$$

The basic principle of dynamic programming is called the principle of optimality. Richard Bellman developed it, and it describes the property of an optimal policy. The principle of optimality for this problem is outlined below.

Note that in the following computations, we use the simplified notation for $e(s,X_s,\zeta)$, $\sigma_2(s,X_s,\zeta)\,and\,\sigma_1(s,X_s,\zeta)$, i.e., we use $e,\sigma_2\ and\,\sigma_1$.

Theorem 1 (Principle of optimality) Let $(t,x) \in [0,T]\times \mathcal{R}$, $\Delta t > 0$ and $t + \Delta t < T$. The value function $V(t,x)$ can be expressed as

$$V(t,x) = \sup_{\zeta} E\left[\int_t^{t+\Delta t} f(s,X_s,\zeta)ds + V(t+\Delta t, x+\Delta X_t)\right] \tag{6}$$

given that $x + \Delta X_t = X_{t+\Delta t}$.

Proof 1 Let $\delta = t + \Delta t, \mu = x + \Delta X_t$ and

$$V^*(t,x) = \sup_{\zeta} E\left[\int_t^{\partial} f(s, X_s, \zeta) ds + V(\delta, \mu)\right].$$

From the definition of $V(t,x)$, we have

$$V(t,x) \geq E\left[\int_t^{\delta} f\left(s, X_s, \zeta|_{[t,\delta)}\right) ds + \int_{\delta}^{T} f\left(s, X_s, \zeta|_{[\delta,T]}\right) ds + Q(X_T, T)\right] \tag{7}$$

given a control process ζ. The values of the quantity that the decision-maker controls are represented by ζ. On the intervals $[t,\delta)$ and $[\delta, T]$, these values can be expressed as $\zeta|_{[t,\delta)}$ and $\zeta|_{[\delta,T]}$ in that order. The integrals represented by

$$\int_t^{\delta} f\left(s, X_s, \zeta|_{[t,\delta)}\right) ds$$

and

$$\int_{\delta}^{T} f\left(s, X_s, \zeta|_{[\delta,T]}\right) ds$$

are autonomous from each other since uncertain stochastic processes

$$G\_\{t\} = (dB_t, dC_t)(s \in [t, \delta))$$

and

$$G\_\{t\} = (dB_t, dC_t)(s \in [t, \delta])$$

are also autonomous from each other. Applying theorem 5 in [35] to Equation (6), we get

$$V(t,x) \geq E\left[\int_t^{\delta} f\left(s, X_s, \zeta|_{[t,\delta)}\right) ds\right] + E\left[\int_{\delta}^{T} f\left(s, X_s, \zeta|_{[\delta,T]}\right) ds + Q(X_T, T)\right]. \tag{8}$$

If we take the supremum of the right-hand side in Equation (7) with respect to $\zeta|_{[t,\delta)}$ and $\zeta|_{[\delta,T]}$, it can be concluded that $V(t,x) \geq V^*(t,x)$. However,

$$E\left[\int_t^{T} f(s, X_s, \zeta) ds + Q(X_t, T)\right] = E\left[\int_t^{\delta} f\left(s, X_s, \zeta|_{[t,\delta)}\right) ds\right] +$$

$$+ E\left[E\left[\int_{\delta}^{T} f\left(s, X_s, \zeta|_{[\delta,T]}\right) ds + Q(X_T, T)\right]\right] \leq E\left[\int_t^{\delta} f(s, X_s, \zeta) ds + V(\delta, \mu)\right] \leq V^*(t,x).$$

This means $V(t,x) \leq V^*(t,x)$, thus $V(t,x) = V^*(t,x)$, which concludes the proof. For an uncertain random optimal control problem in Equation (4), the optimality equation is presented in the following theorem.

Theorem 2 (Equation of optimality) If $V(t,x) : [0,\ T] \times \mathcal{R} \to \mathcal{R}$ is a twice continuously differentiable function,

$$-V_t(t,x) = \sup_{\zeta}\left[f(t,x,\zeta) + V_x(t,x) e(t,x,\zeta) + \frac{1}{2} V_{xx}(t,x) \sigma_1^2(t,x,\zeta)\right].$$

Proof 2 If $\Delta t > 0$,

$$\int_t^{\delta} f(s, X_s, \zeta) ds = f(t,x,\zeta)\Delta t + o(\Delta t).$$

Using the Taylor series technique, the result is

$$V(\delta,\mu)=V(t,x)+V_t(t,x)\Delta t+V_x(t,x)\Delta X_t+$$
$$+\frac{1}{2}V_{tt}(t,x)(\Delta t)^2+\frac{1}{2}V_{xx}(t,x)(\Delta X_t)^2+V_{xt}(t,x)\Delta t\Delta X_t+o(\Delta t).$$

From Equation (5), we get

$$V(t,x)=\sup_{\zeta}E\left[\int_t^{\delta}f(s,X_s,\zeta)ds+V(\delta,\mu)\right]=\sup_{\zeta}\{f(t,x,\zeta)\Delta t+V(t,x)+V_t(t,x)\Delta t+E\{V_x(t,x)\Delta X_t+$$
$$+\frac{1}{2}V_{tt}(t,x)(\Delta t)^2+\frac{1}{2}V_{xx}(t,x)(\Delta X_t)^2+V_{xt}(t,x)\Delta t\Delta X_t\}+o(\Delta t)\}.$$

Collecting like terms, we have

$$V(t,x)-V(t,x)=\sup_{\zeta}\{f(t,x,\zeta)\Delta t+V_t(t,x)\Delta t+E\left\{\begin{matrix}V_x(t,x)\Delta X_t+\frac{1}{2}V_{tt}(t,x)(\Delta t)^2+\\+\frac{1}{2}V_{xx}(t,x)(\Delta X_t)^2\ +V_{xt}(t,x)\Delta t\Delta X_t\end{matrix}\right\}+o(\Delta t)\}=0.$$

From Equation (2), we have

$$(\Delta X_t)^2=e^2(\Delta t)^2+\sigma_2{}^2(\Delta C_t)^2+\sigma_1{}^2(\Delta B_t)^2+$$
$$+2\sigma_2{}^2\sigma_1{}^2\Delta C_t\Delta B_t+2e\sigma_2\Delta t\Delta C_t+2e\sigma_1\Delta t\Delta B_t,$$
$$\Delta t\Delta X_t=e(\Delta t)^2+\sigma_2\Delta t\Delta C_t+\sigma_1\Delta t\Delta B_t.$$

and

$$\Delta X_t=e(\Delta t)+\sigma_2\Delta C_t+\sigma_q\Delta B_t.$$

Replacing $(\Delta t)^2,(\Delta C_t)^2,\Delta t\Delta B_t,\Delta t\Delta C_t$ *and* $\Delta C_t\Delta B_t$ *by* 0 *and setting* $(\Delta B_t)^2=\Delta t$ *in the equations of* $(\Delta X_t)^2$ *and* $\Delta t\Delta X_t$ *yields* $\Delta t\Delta X_t=0$ *and* $(\Delta X_t)^2=\sigma_1{}^2\Delta t$, *respectively*. This means

$$-V_t(t,x)\Delta t=\sup_{\zeta}\{f(t,x,\zeta)\Delta t+E\{V_x(t,x)\Delta X_t+\frac{1}{2}V_{tt}(t,x)(\Delta t)^2=$$
$$=\sup_{\zeta}\{f(t,x,\zeta)\Delta t+V_x(t,x)E[\Delta X_t]+\frac{1}{2}V_{tt}(t,x)(\Delta t)^2+$$
$$+\frac{1}{2}V_{xx}(t,x)E\left[(\Delta X_t)^2\right]+V_{xt}(t,x)E\left[\Delta t\Delta X_t\right]+o(\Delta t)\}=$$
$$=\sup_{\zeta}\{f(t,x,\zeta)\Delta t+V_x(t,x)e\Delta t+E[\sigma_2\Delta C_t+\sigma_1\Delta B_t]+\frac{1}{2}V_{xx}(t,x)E\left[\sigma_1{}^2\Delta t\right]+o(\Delta t)\}=$$
$$=\sup_{\zeta}\{f(t,x,\zeta)\Delta t+V_x(t,x)e\Delta t+\frac{1}{2}V_{xx}(t,x)\sigma_1{}^2\Delta t\}=$$
$$=\Delta t\sup_{\zeta}\ \{f(t,x,\zeta)+V_x(t,x)e+\frac{1}{2}V_{xx}(t,x)\sigma_1{}^2\}$$

since

$$E[\sigma_2\Delta C_t+\sigma_1\Delta B_t]=0$$

and

$$E\left[\sigma_1{}^2\Delta t\right]=\sigma_1{}^2\Delta t\,.$$

If we divide both sides by Δt, we get

$$-V_t(t,x)=\sup_{\zeta}\{f(t,x,\zeta)+V_x(t,x)e(t,x,\zeta)+\frac{1}{2}V_{xx}(t,x)\sigma_1^2(t,x,\zeta)\}.$$

Thus, the theorem has been proven.

The following section applies the above concepts to solve a portfolio selection problem in uncertain random markets.

Dynamic programming with applications to the portfolio selection model in uncertain random environments

Consider an uncertain random financial market with bond price S_t^* and stock price S_t described by

$$\begin{cases} dS_t^*=rS_t^*\,dt \\ dS_t=eS_tdt+\sigma_2S_tdC_t+\sigma_1S_tdB_t \end{cases}$$

where the risk-less interest rate is given by r, the mean rate of return for the risky asset is e,σ_2 is uncertain variance, and σ_1 is stochastic variance.

If X_t is the investor's wealth at time t, and the investor allocates $1-\eta(t)$ to represent the fraction of a sure asset, the fraction $\eta(t)$ caters for the risky asset. Let $Z_t=X^{\eta(t)}$ be an uncertain random wealth process for an investment strategy ζ. Suppose $Z_t=X^{\eta(t)}$, the wealth process given the risky asset return

$$\frac{dS_t}{S_t}=edt+\sigma_2dC_t+\sigma_1dB_t$$

in the interval $(t,t+dt]$ is

$$\begin{aligned} dZ_t &= r(1-\eta(t))Z_tdt+\frac{dS_t}{S_t}\eta(t)Z_t= \\ &= Z_t\left[r\eta(t)+e(1-\eta(t))\right]dt+ \\ &+[\sigma_2Z_tdC_t+\sigma_1Z_tdB_t]\eta(t). \end{aligned}$$

Choosing the power utility function similar to equation (11.2.52) in [21], and assuming that there is no bequest, the portfolio selection model for an investor who is concerned with maximizing the expected utility on an infinite time interval is given by

$$\begin{cases} V(t,x)\equiv\max_{\eta(t)}\left[\int_0^T\exp(-\gamma t)\frac{(Z_t)^\lambda}{\lambda}dt\right] \\ \text{subject to} \\ dZ_t=Z_t\left[r\eta(t)+e(1-\eta(t))\right]dt+[\sigma_2Z_tdC_t+\sigma_1Z_tdB_t]\eta(t). \end{cases} \tag{9}$$

The value function in equation (9) is obtained from equation (5), when $Q(X_T,T)$ *is assumed to be* 0. Under equation (9), γ is taken to be greater than 0, that is $\gamma>0$, and λ is considered to lie between 0 and 1, i.e., $0<\lambda<1$. Applying the equation in Theorem 3.2, we get

$$-V_t(t,x)=\sup_{\zeta}\left[\exp(-\gamma t)\frac{z^\lambda}{\lambda}+z\left[r\eta(t)+e(1-\eta(t))\right]V_z+\frac{1}{2}V_{zz}\sigma_1^2z^2\eta(t)^2\right]. \tag{10}$$

Let

$$\begin{aligned} L(\eta(t)) &= \exp(-\gamma t)\frac{z^\lambda}{\lambda}+z\left[r\eta(t)+e(1-\eta(t))\right]V_z+\frac{1}{2}V_{zz}\sigma_1^2z^2\eta(t)^2= \\ &= \exp(-\gamma t)\frac{z^\lambda}{\lambda}+z\left[e+(r-e)\eta(t)\right]V_z+\frac{1}{2}V_{zz}\sigma_1^2z^2\eta(t)^2. \end{aligned}$$

An optimal $\eta(t)$ satisfies

$$\frac{\partial L(\eta(t))}{\partial \eta(t)} = (r-e)zV_z + V_{zz}\sigma_1{}^2 z^2 \eta(t) = 0.$$

Thus

$$\eta(t) = -\frac{(r-e)V_z}{z\sigma_1{}^2 V_{zz}}. \tag{11}$$

Substituting $\eta(t)$ into Equation (10), we get

$$-V_t = \exp(-\gamma t)\frac{z^\lambda}{\lambda} + z\left[e + (r-e)\left(-\frac{(r-e)V_z}{z\sigma_1{}^2 V_{zz}}\right)\right]V_z +$$

$$+\frac{1}{2}V_{zz}\sigma_1{}^2 z^2\left(-\frac{(r-e)V_z}{z\sigma_1{}^2 V_{zz}}\right)^2 =$$

$$= \exp(-\gamma t)\frac{z^\lambda}{\lambda} + zeV_z - \left(\frac{(r-e)^2 V_z{}^2}{2y_1{}^2 V_{zz}}\right). \tag{12}$$

If we make a conjuncture that $V(t,z) = kz^\lambda \exp(-\gamma t)$, then we get

$$V_t = -k\gamma z^\lambda \exp(-\gamma t),$$

$$V_z = k\lambda z^{\lambda-1} \exp(-\gamma t)$$

and

$$V_{zz} = k\lambda(\lambda-1)z^{\lambda-2} \exp(-\gamma t).$$

After multiplying Equation (12) by $\exp(\gamma t)$ throughout, we get

$$-V_t \exp(\gamma t) = \frac{z^\lambda}{\lambda} + zeV_z \exp(\gamma t) - \left(\frac{(r-e)^2 V_z{}^2}{2\sigma_1{}^2 V_{zz}}\right)\exp(\gamma t). \tag{13}$$

Substituting $V_t, Vzz,$ and V_z in Equation (13), we obtain

$$k\gamma z^\lambda = \frac{z^\lambda}{\lambda} + ek\lambda z^\lambda - \left(\frac{(r-e)^2 k\lambda z^\lambda}{2\sigma_1{}^2(\lambda-1)}\right). \tag{14}$$

Dividing by kz^λ, we get

$$\gamma = \frac{1}{k\lambda} + e\lambda - \left(\frac{(r-e)^2 \lambda}{2\sigma_1{}^2(\lambda-1)}\right).$$

If V_t, V_{zz} and V_z are substituted in Equation (11), the fraction of investment in a risky asset that is optimal becomes

$$\eta(t) = \frac{(r-e)}{\sigma_1^{\,2}(\lambda-1)}.$$

The above optimal fraction of investment is independent of the total wealth. This fraction of investment in a risky asset that is optimal is the same as the one obtained by [21].

Conclusions

This study analyzed an uncertain stochastic optimal control problem premised on the notion of an uncertain stochastic process. Further, the Bellman's principle of optimality in dynamic programming was implemented to deduce the principle of optimality, and then the resulting Hamilton-Jacobi-Bellman equation (the equation of optimality in uncertain stochastic optimal control) was applied to solve a proposed portfolio selection problem.

The results of this study indicate that the dynamic programming principle for optimal control of USDEs can be applied in optimal portfolio selection. Also, the study results show that the optimal fraction of investment is independent of wealth. The results are valuable for solving the optimal portfolio selection problem in Itô-Liu financial markets. The main conclusion of this study is that, in Itô-Liu financial markets, the dynamic programming principle for optimal control of USDEs can be applied in solving the optimal portfolio selection problem. Even though this study has produced interesting results, there is room for extension. The study can be extended by solving the optimal portfolio selection model with jumps for Itô-Liu financial markets.

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