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Proximity of Bachelier and Samuelson Models for Different Metrics

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ABSTRACT

This paper proposes a method of comparing the prices of European options, based on the use of probabilistic metrics, with respect to two models of price dynamics: Bachelier and Samuelson. In contrast to other studies on the subject, we consider two classes of options: European options with a Lipschitz continuous payout function and European options with a bounded payout function. For these classes, the following suitable probability metrics are chosen: the Fortet-Maurier metric, the total variation metric, and the Kolmogorov metric. It is proved that their computation can be reduced to computation of the Lambert in case of the Fortet-Mourier metric, and to the solution of a nonlinear equation in other cases. A statistical estimation of the model parameters in the modern oil market gives the order of magnitude of the error, including the magnitude of sensitivity of the option price, to the change in the volatility.

Keywords: Bachelier model; Samuelson model; option pricing; probabilistic metrics; sensitivity; volatility

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Близость моделей Башелье и Самуэльсона для различных метрик

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АННОТАЦИЯ

В статье представлен метод сравнения цен европейских опционов, основанный на использовании вероятностных метрик, применительно к двум моделям динамики цен — Башелье и Самуэльсона. В отличие от других работ на данную тему, рассматриваются классы опционов, а именно европейские опционы с функцией выплат, удовлетворяющих условию Липшица, а также европейские опционы с ограниченной функцией выплат. Для данных классов выбираются подходящие вероятностные метрики: метрика Форте-Мурье, метрика полной вариации и метрика Колмогорова. Мы доказали, что их вычисление сводится к вычислению W-функции Ламберта в случае метрики Форте-Мурье и к решению некоторого нелинейного уравнения в остальных случаях. Статистическая оценка параметров моделей на современном нефтяном рынке указывает на порядок величины погрешности, включая величину чувствительности цены опциона к изменению показателя волатильности.

Ключевые слова: модель Башелье; модель Самуэльсона; ценообразование опционов; вероятностные метрики; чувствительность; волатильность

1 Introduction

Description of Models and Motivation for the Study

In this study, the simplest continuous-time financial market models are considered. The movement of prices $(X_t)_{t\in[0,T]}$ of an asset in the market is described in the framework of the Bachelier model (Bachelier, 1900), using the stochastic Brownian motion process:

$$X_t^B = X_0\left(1+\alpha t+\sigma_B W_t\right), t\in\left[0,T\right], \#\left(1\right)$$

where $(W_t)_{t\in[0,T]}$ is the Wiener process, $\alpha\in\mathbb{R}, \sigma_B>0$.

The model proposed by Samuelson[1] (1965) uses geometric (economic) Brownian motion to describe the price dynamics:

$$X_t^S = X_0\exp\left[\gamma t+\sigma_S W_t\right], t\in\left[0,T\right], \#\left(2\right)$$

where $\gamma\in\mathbb{R}, \sigma_S>0$.

In both models, the volatilities σ_B and σ_S are chosen so that they have the dimension $[\text{time}]^{-1/2}$ and the linear trend α and exponential trend γ have the dimension $[\text{time}]^{-1}$.

Hereafter, the prices considered are assumed to be discounted, which is equivalent to a zero risk-free interest rate.

The Black-Scholes (1973) and Merton (1973) option pricing model is based on the Samuelson model (describing price dynamics in the market) and is the most popular in practice. Similarly, for the options on futures Black's (1976) pricing model is based on Samuelson's model.

Bachelier (1900) not only described the dynamics of prices but also built a model of option pricing. However, Samuelson (1965) noted that the stock prices should not be negative; thus, Bachelier's model has not been widely used in practice. Nevertheless, for short-term options, the Bachelier model can better fit the real market data than the Black-Scholes–Samuelson model (e.g., Versluis (2006)). Note that the Bachelier model and its modifications have been applied to modern works on mathematical finance. For example, the Bachelier model and its modification with an absorption screen was used by Glazyrina and Melnikov (2020) for pricing life insurance policies with an invested stock index option, and Melnikov and Wan (2021) compared this model with the Bachelier and Samuelson models.

An unprecedented event occurred on April 20, 2020, when West Texas Intermediate (WTI) crude oil futures prices (the benchmark for U.S. crude oil prices) reached negative levels (see CFTC Interim Staff Report, Trading in NYMEX WTI Crude Oil Futures Contract Leading up to, on, and around April 20, 2020). Fuel supply has far exceeded the demand due to the coronavirus pandemic. Due to overproduction, the storage tanks were so full that it would have been difficult to find room for new oil if the future contracts had been brought to delivery. Because the May contract expired on April 21, market participants with long positions did not want to take delivery of oil (which no one needed at that point in time) and incur storage costs and opted to lock in such large losses by entering into offset deals that the prices turned negative. As of April 22, 2020, the Chicago Mercantile Exchange (CME) switched to the Bachelier pricing model for the options on futures for several energy commodities[2] to account for the possibility of negative prices.

In this regard, it is interesting to compare the prices of derivative financial instruments obtained using the above-described models. Schachermayer and Teichmann (2005) proved the following estimation for the price difference of a call option "at the money" with an expiration at the moment T:

$$0\le C_B - C_S \le \frac{X_0}{12\sqrt{2\pi}}(\sigma\sqrt{T})^3.$$

Here, $\sigma_B=\sigma_S=\sigma$ and C_B, C_S denote the option prices in the Bachelier and Samuelson models, respectively. Both processes (1) and (2) are diffusion processes; thus, the Bachelier and Samuelson models are clearly close in case of small (and equal) values of integral volatility $\sigma_B\sqrt{T}=\sigma_S\sqrt{T}=\sigma\sqrt{T}$. Meanwhile, the Samuelson model is close[3] to the Bachelier model with a linear trend $\gamma+\frac{1}{2}\sigma^2$.

Grunspan (2011) obtained an asymptotic relation between implicit volatilities for normal and lognormal models at $T\to 0$ and compared the sensitivities (greeks) for call options. The differences in option pricing obtained using the Bachelier and Samuelson models are detailed in Thomson (2016).

Another question is for what values of $\sigma_B\sqrt{T}$ and $\sigma_S\sqrt{T}$ models can be considered close? We are interested in the problem of comparing the prices of a European option with an arbitrary payoff function $f(\cdot)$ that belongs to a specific class of functions and depends only on the price X_T of the underlying asset at the time of expiration T. For each of the models (1) and (2), there exists a single equivalent risk-neutral (martingale) measure. The option price $P(f,T)$ with payout function $f(\cdot)$ and time to expiration T is determined as the mathematical expectation relative to the corresponding risk-neutral measure[4]:

$$P(f,T)=\mathbb{E}^{*}f(X_T).$$

The processes given by relations (1) and (2) are martingales if and only if

$$\alpha=0,\gamma=-\frac{\sigma_S^2}{2}.\#(3)$$

Therefore, the difference between the option prices $P_B(f,T)$ and $P_S(f,T)$ in the Bachelier and Samuelson models can be expressed as follows:

$$P_B(f,T)-P_S(f,T)=Ef\left(X_T^B\right)-Ef\left(X_T^S\right),\#(4)$$

where the process parameters are chosen according to (3).

The estimate for the right part of (4) can be obtained by calculating the distance in the Fortet–Mourier metric between the distributions of random variables X_T^B,X_T^S in case of Lipschitz continuity of the payoff function $f(\cdot)$. If the payout function is discontinuous but bounded (e.g., as in the case of a binary option), the total variation metric can be used for the estimation. However, the Kolmogorov metric can also be used to compare the binary option prices; the closeness of distributions under the total variation metric is a very strong assumption, and hence, the corresponding estimate is rougher (but applicable to a broader class of payout functions).

To compare the Bachelier and Samuelson models, it is interesting to find the optimal relation between the volatilities σ_B,σ_S. Optimality is understood as the dependence between these indicators that arises when minimizing the distance between X_T^B and X_T^S in (one or another) probability metric $d(\cdot,\cdot)$.

In this paper, the Fortet–Mourier metric between random variables X_T^B and X_T^S is calculated and the formulae for the total variation metric and Kolmogorov metric are obtained. The dependence of volatilities that minimizes the Fortet-Mourier metric between X_T^B and X_T^S. Using the probability metrics, the estimates for (4) are obtained to analyze the effect of model choice on option price. By constructing confidence intervals for volatilities in the oil market for standard and binary call and put options, we evaluate the error resulting from the approximate measurement of the volatility.

Notation and Definitions

Let S be a metric space with metric $d(\cdot,\cdot)$ and let us denote by $\mathcal{M}(S)$ the set of all signed measures on S and $\mathcal{P}(S)\subset\mathcal{M}(S)$ as the set of all probability measures on S equipped with Borel σ-algebra.

Definition 1. *Let us define a semi-norm in the space $Lip(S)$ of the Lipschitz continuous on S functions as follows:*

$$\|f\|_{Lip}=\sup_{x,y}\frac{|f(x)-f(y)|}{d(x,y)},f(\cdot)\in Lip(S).$$

Definition 2. *In the space $B(S)$ of bounded measurable functions on S, let us define the norm*

$$\|f\|_B=\sup_{x\in S}|f(x)|,f(\cdot)\in B(S).$$

Definition 3. *For $S=\mathbb{R}$ in the space $St(\mathbb{R})$ of piecewise constant functions with finite number of jumps $\Delta_1,\ldots,\Delta_m$, we define a semi-norm*

$$\|f\|_{St}=\sum_{j=1}^{m}|\Delta_j|,f(\cdot)\in St(\mathbb{R}).$$

The introduced semi-norm is a norm in space $St(\mathbb{R})/\mathbb{R}$.

Definition 4. *By the coupling of two random variables X u Y, we call[5] a pair (X',Y') for which the following is true $X'\overset{d}{=}X,Y'\overset{d}{=}Y$. For the monotone coupling of real random variables X u Y with distribution functions $F_X(\cdot),F_Y(\cdot)$, we call a pair of*

$$\left(F_X^{-1}(U),F_Y^{-1}(U)\right),U\sim\mathcal{U}(0,1),$$

where F_X is the distribution function of a random variable X, which is defined as

$$F_X(x)=\mathbb{P}(X<x),$$

and F^{-1} is the generalized inverse function of the monotonically non-decreasing left-continuous function, defined via the relation[6]

$$F^{-1}(y)=\inf\{x\in\mathbb{R}:F(x)\geq y\}=$$
$$=\sup\{x\in\mathbb{R}:F(x)<y\},y\in(0,1).$$

Let $\delta(\cdot,\cdot)$ be a metric in the space of random variables taking values in S, defined on pairs of (X,Y) of random variables, with a common probability space.

Definition 5. *The minimal metric with respect to $\delta(\cdot,\cdot)$ is the metric*

$$\hat{\delta}(X,Y)=\inf\left\{\delta(X',Y'):X'\overset{d}{=}X,Y'\overset{d}{=}Y\right\}.$$

Note that $\delta'(\cdot,\cdot)$ is therefore a metric in the space of distributions and does not depend on the joint distribution of X and Y.

Let $\mathcal{F}$ be a set of measurable functions $f:S\to\mathbb{R}$. Then, for each signed measure μ on S such that $\int_S|f||d\mu|<\infty$ for all $f\in\mathcal{F}$, the following semi-norm can be defined:

$$\|\mu\|^*_{\mathcal{F}}=\sup_{f\in\mathcal{F}}\left|\int_S fd\mu\right|.$$

Denote $\mathcal{M}_{\mathcal{F}}=\{\mu\in\mathcal{M}(S):\|\mu\|^*_{\mathcal{F}}<\infty\}$.

Definition 6. *We can say that on $\mathcal{M}_{\mathcal{F}}$ the dual semimetric if*

$$d_{\mathcal{F}}(\mu,\nu)=\|\mu-\nu\|^*_{\mathcal{F}}.$$

In particular, for the probabilistic measures $\mathcal{P}_{\mathcal{F}}=\mathcal{M}_{\mathcal{F}}\bigcap\mathcal{P}(S)$,

$$d_{\mathcal{F}}(X,Y)=\sup_{f\in\mathcal{F}}\left|\mathbb{E}f(X)-\mathbb{E}f(Y)\right|.$$

Let $(S,\mathcal{B})$ be a measurable space.

Definition 7. *The total variation norm for a signed measure μ is defined as*

$$\|\mu\|_{TV}=\sup\left\{\int_S fd\mu:f\in B(S),\|f\|_B\leq 1\right\}.$$

Definition 8. *A total variation metric is a probability metric*

$$d_{TV}(Q_1,Q_2)=\|Q_1-Q_2\|_{TV}.$$

If distributions Q_1,Q_2 are absolutely continuous with respect to the measure μ and have Radon–Nikodym densities $p_1(\cdot),p_2(\cdot)$, then

$$d_{TV}(Q_1,Q_2)=\int_S\left|p_1(x)-p_2(x)\right|\mu(dx)=$$
$$=2\int_S(p_1(x)-p_2(x))^+\mu(dx),\#(5)$$

where $a^+=\max(a,0)$.

Definition 9. *If $S=\mathbb{R}$, then the Kolmogorov metric*[7] *is*

$$d_K(X,Y)=\sup_{x\in\mathbb{R}}\left|F_X(x)-F_Y(x)\right|.$$

Definition 10. *The Fortet-Mourier metric*[8] *is the probability metric*

$$d_{FM}(X,Y)=\sup_{\|f\|_{Lip}\leq 1}\left|\mathbb{E}f(X)-\mathbb{E}f(Y)\right|.$$

There is also an equivalent representation of this metric:

$$d_{FM}(X,Y)=\min\left\{\mathbb{E}d(X',Y'):X'\overset{d}{=}X,Y'\overset{d}{=}Y\right\}.\#(6)$$

The proof of equivalence of the definitions can be found in Rachev, Klebanov, Stoyanov, and Fabozzi (2013).

It has been shown (e.g., Bogachev (2007)) that in case of $S=\mathbb{R}$, the minimum value in (6) is attained on the monotone coupling

$$\left(F_X^{-1}(U),F_Y^{-1}(U)\right),U\sim\mathcal{U}(0,1).$$

Remark 1. *The Fortet-Mourier metric allows one to derive an upper estimate of (4) in the case of Lipschitz continuity of $f(\cdot)$, for example, if $f(\cdot)$ is piecewise linear (which corresponds to the portfolio of call and put options). It is also possible to estimate (4) by using the total variation metric if the function $f(\cdot)$ is bounded. Even if the payout function is neither Lipschitz continuous nor bounded (e.g., if it corresponds to a portfolio of binary and call options), it can most likely be represented as a sum of ones, as in practice, the payout functions usually do not grow faster than linear ones. The Kolmogorov metric provides a more accurate estimate than the total variation metric; however, it is only applicable to piecewise constant payout functions corresponding to a portfolio composed of binary options.*

Definition 11. Lambert W *function is a complex-valued function $W:\mathbb{C}\to\mathbb{C}$, defined as a solution of the equation $z=W(z)e^{W(z)},z\in\mathbb{C}$.*

$W(\cdot)$ cannot be expressed in elementary functions. We are only interested in its two branches, $W_0(z), W_{-1}(z)$, at $z \in (-e^{-1}, 0)$ (Fig. 1), which correspond to the real solutions of the equation

$$xe^x = z, z \in (-e^{-1}, 0).$$

The definition and notation are taken from Corless, Gonnet, Hare, Jeffrey, and Knuth (1996).

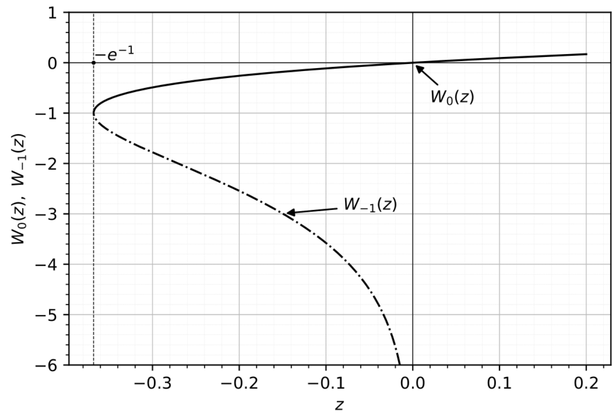

Figure 1. Real-valued branches of Lambert W -function

Source: The authors.

2 Main Results

Let us show how one can obtain the estimates for (4) by using the introduced probability metrics. Let, as mentioned above, $P_B(f,T), P_S(f,T)$ stand for the prices of European options with payoff function $f(\cdot)$ and time to expiration T in the Bachelier and Samuelson models, respectively. Then, the following estimates are true:

If $f(\cdot) \in Lip(\mathbb{R})$, then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{Lip}\, d_{FM}\left(X_T^B, X_T^S\right). \#(7)$$

If $f(\cdot) \in B(\mathbb{R})$, then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{B}\, d_{TV}\left(X_T^B, X_T^S\right). \#(8)$$

If $f(\cdot) \in St(\mathbb{R})$, then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{St}\, d_{K}\left(X_T^B, X_T^S\right). \#(9)$$

Indeed, the price of a European option is defined in the Bachelier and Samuelson models as a mathematical expectation of the payout function relative to the risk-neutral measure:

$$P_B(f,T) = \mathbb{E}f\left(X_T^B\right), P_S(f,T) = \mathbb{E}f\left(X_T^S\right),$$

where the processes X_t^B, X_t^S are martingales, i.e., $\alpha = 0, \gamma = -\frac{\sigma_S^2}{2}$.

Then,

$$\left|P_B(f,T) - P_S(f,T)\right| = \left|\mathbb{E}\left(f\left(X_T^B\right) - f\left(X_T^S\right)\right)\right|.$$

1. In case of Lipschitz continuity of $f(\cdot)$,

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{Lip} \sup_{\|g\|_{Lip} \le 1} \left|\mathbb{E}g\left(X_T^B\right) - \mathbb{E}g\left(X_T^S\right)\right| = \| f \|_{Lip}\, d_{FM}\left(X_T^B, X_T^S\right)$$

2. If $f(\cdot)$ is bounded, then

$$\left|P_B(f,T) - P_S(f,T)\right| = \left|\int_{\mathbb{R}} f(x)\left(p_{X_T^B}(x) - p_{X_T^S}(x)\right)dx\right| \le$$

$$\le \| f \|_B \int_{\mathbb{R}} \left|p_{X_T^B}(x) - p_{X_T^S}(x)\right| dx = \| f \|_B\, d_{TV}\left(X_T^B, X_T^S\right).$$

Here, $p_{X_T^B}(\cdot), p_{X_T^S}(\cdot)$ denote the densities of random variables X_T^B, X_T^S.

3. The function $f(\cdot) \in St(\mathbb{R})$ can be represented as

$$f(X_T) = f(-\infty) + \sum_{j=1}^{m} f_j(X_T), f_j(x) = \Delta_j \mathbb{I}_{x>K_j}.$$

For each function, $f_j(\cdot)$ it is true that

$$\left|P_B(f_j,T) - P_S(f_j,T)\right| = \left|\Delta_j\right| \left|F_{X_T^S}(K_j) - F_{X_T^B}(K_j)\right| \le \left|\Delta_j\right| d_K\left(X_T^B, X_T^S\right).$$

$$\left|P_B(f,T) - P_S(f,T)\right| \le \sum_{j=1}^{m} \left| P_B(f_j,T) - P_S(f_j,T)\right| \le \sum_{j=1}^{m} \left|\Delta_j\right| d_K\left(X_T^B, X_T^S\right) =$$
$$= \| f \|_{St} \, d_K\left(X_T^B, X_T^S\right).$$

Note 2: *If the payout function can be represented as*

$$f(\cdot) = f_1(\cdot) + f_2(\cdot) + f_3(\cdot), f_1(\cdot) \in Lip(\mathbb{R}), f_2(\cdot) \in B(\mathbb{R}), f_3(\cdot) \in St(\mathbb{R}), \#(10)$$

then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f_1 \|_{Lip} \, d_{FM}\left(X_T^B, X_T^S\right) + \| f_2 \|_B \, d_{TV}\left(X_T^B, X_T^S\right) + \| f_3 \|_{St} \, d_K\left(X_T^B, X_T^S\right). \#(11)$$

The representation (10) is obviously not unique. Moreover, $f_3(\cdot)$ is unnecessary as soon as any piecewise constant function with a finite number of jumps is bounded. Nevertheless, a proper choice of functions $f_1(\cdot), f_2(\cdot)$ и $f_3(\cdot)$ in expansion (10) can significantly improve the estimate (11).

The following statements provide methods of calculation of the metrics appearing in (7)–(9).

Finding $d_{FM}\left(X_t^B, X_t^S\right)$ is reduced to the calculation of the metric between random variables $\xi \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ and $\eta \sim \mathcal{LN}\left(\mu_2, \sigma_2^2\right)$ that have normal and lognormal distributions. The value of this metric is given by the following theorem.

Theorem 1
Let $\xi \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right), \eta \sim \mathcal{LN}\left(\mu_2, \sigma_2^2\right)$. Then, under the condition $\ln\left(\frac{\sigma_2}{\sigma_1}\right) + \mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1 + 1 < 0, \#(*)$,

the metric can be found with the formula

$$d_{FM}(\xi,\eta) = \mu_1\left(2\left[\Phi(k_2) - \Phi(k_1)\right] - 1\right) + 2\sigma_1\left(\phi(k_1) - \phi(k_2)\right) +$$
$$+ \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right]\left(1 - 2\left[\Phi(k_2 - \sigma_2) - \Phi(k_1 - \sigma_2)\right]\right), \quad \#(12)$$

where $\Phi(\cdot)$ is a cumulative distribution function of the standard normal distribution, $\phi(\cdot)$ is the density of the standard normal distribution, and k_1 and k_2 are equal to

$$k_1 = -\frac{\mu_1}{\sigma_1} - \frac{1}{\sigma_2} W_0\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1\right]\right),$$
$$k_2 = -\frac{\mu_1}{\sigma_1} - \frac{1}{\sigma_2} W_{-1}\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1\right]\right). \quad \#(13)$$

If condition (*) is not satisfied, then

$$d_{FM}(\xi,\eta) = -\mu_1 + \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right]. \#(14)$$

Corollary 1. *When trends and volatilities are chosen such that processes (1) and (2) are martingales (i.e., relation (3) is satisfied), the formula for the metric between distributions of the random variables* X_t^B, X_t^S *can be expressed as*

$$d_{FM}\left(X_t^B, X_t^S\right) = 2X_0(\left[\Phi(k_2)-\Phi(k_1)\right]-\left[\Phi(k_2-\sigma_2)-\Phi(k_1-\sigma_2)\right] \\ -\left[\phi(k_2)-\phi(k_1)\right]), \#(15)$$

where $\sigma_1 = \sigma_B\sqrt{t}, \sigma_2 = \sigma_S\sqrt{t}$ *denote the integral volatilities, and* k_1, k_2 *are calculated as follows:*

$$\begin{aligned} k_1 &= -\frac{1}{\sigma_1}-\frac{1}{\sigma_2}W_0\left(-\frac{\sigma_2}{\sigma_1}\exp\left[-\frac{\sigma_2^2}{2}-\frac{\sigma_2}{\sigma_1}\right]\right), \\ k_2 &= -\frac{1}{\sigma_1}-\frac{1}{\sigma_2}W_{-1}\left(-\frac{\sigma_2}{\sigma_1}\exp\left[-\frac{\sigma_2^2}{2}-\frac{\sigma_2}{\sigma_1}\right]\right). \end{aligned} \#(16)$$

The following theorem answers the question about the optimal relation between σ_B and σ_S minimize the Fortet–Mourier metric in the risk-neutral case.

Theorem 2
For fixed σ_2, *the minimum of expression (15) is attained at*

$$\sigma_1^* = \frac{\sigma_2\sqrt{1-e^{-\sigma_2^2}}}{\frac{\sigma_2^2}{2}+\ln\left(1+\sqrt{1-e^{-\sigma_2^2}}\right)}.$$

For fixed σ_1, *the minimum in (15) is attained at* σ_2^*, *which is a solution of the equation* $k_1+k_2=2\sigma_2$, *where* k_1, k_2 *are determined from (16).*

The calculation of the total variation metric and the Kolmogorov metric between X_T^B and X_T^S can be reduced to solving a nonlinear equation. This result is formulated in Theorem 3.

Theorem 3
L $\xi \sim \mathcal{N}\left(\mu_1,\sigma_1^2\right), \eta \sim \mathcal{LN}\left(\mu_2,\sigma_2^2\right)$, *and* $\mu_1 = 1, \mu_2 = -\frac{\sigma_2^2}{2}$. *Then,*

$$d_{TV}(\xi,\eta) = 2\left[\left(F_\xi(x_1)-F_\eta(x_1)\right)+\left(F_\xi(x_3)-F_\eta(x_3)\right)-\left(F_\xi(x_2)-F_\eta(x_2)\right)\right], \#(17)$$

$$d_K(\xi,\eta) = \max_{i=1,2,3}\left|F_\xi(x_i)-F_\eta(x_i)\right|, \#(18)$$

where $x_1 \le x_2 \le x_3$ *are the roots of the equation*

$$\left(x^2-2x\right)-\left(\frac{\sigma_1}{\sigma_2}\right)^2(\ln x)^2-3\sigma_1^2\ln x+1-\frac{\sigma_1^2\sigma_2^2}{4}-2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right)=0, \#(19)$$

and the cumulative distribution functions have the form $F_\xi(x)=\Phi\left(\frac{x-\mu_1}{\sigma_1}\right), F_\eta(x)=\Phi\left(\frac{\ln(x)-\mu_2}{\sigma_2}\right)\mathbb{I}_{x\ge 0}$.

Corollary 2. *According to Definitions* 8 *and* 9,

$$d_{TV}\left(X_T^B, X_T^S\right) = d_{TV}\left(\frac{X_T^B}{X_0},\frac{X_T^S}{X_0}\right), d_K\left(X_T^B, X_T^S\right) = d_K\left(\frac{X_T^B}{X_0},\frac{X_T^S}{X_0}\right).$$

In the risk-neutral case, $\frac{X_T^B}{X_0} \sim \mathcal{N}\left(1, \sigma_S^2 T\right), \frac{X_T^S}{X_0} \sim \mathcal{LN}\left(-\frac{\sigma_S^2 T}{2}, \sigma_S^2 T\right)$, *and the metrics are calculated by Theorem* 3 *by taking into account that* $\sigma_1 = \sigma_B \sqrt{T}, \sigma_2 = \sigma_S \sqrt{T}$.

3 Proofs of Theorems

Proof of Theorem 1

The cumulative distribution functions of ξ, η are

$$F_\xi(x) = \Phi\left(\frac{x-\mu_1}{\sigma_1}\right), F_\eta(x) = \Phi\left(\frac{\ln(x)-\mu_2}{\sigma_2}\right)\mathbb{I}_{x\geq 0}.$$

Then, their inverse functions can be expressed as

$$F_\xi^{-1}(u) = \mu_1 + \sigma_1 \Phi^{-1}(u), F_\eta^{-1}(u) = e^{\mu_2 + \sigma_2 \Phi^{-1}(u)}.$$

As the minimum in (6) is attained on the monotone coupling, we obtain

$$d_{FM}(\xi,\eta) = \mathbb{E}\left|(\mu_1 + \sigma_1 Z) - e^{\mu_2 + \sigma_2 Z}\right|, Z = \Phi^{-1}(U) \sim \mathcal{N}(0,1).$$

The expectation is considered here with respect to the measure $\mathbb{P}_Z$ induced by a random variable Z .
Let us divide the space of elementary events into three disjoint sets:

$$D_1 = \left\{\omega : \mu_1 + \sigma_1 Z > e^{\mu_2 + \sigma_2 Z}\right\},$$

$$D_2 = \left\{\omega : \mu_1 + \sigma_1 Z < e^{\mu_2 + \sigma_2 Z}\right\},$$

$$D_3 = \left\{\omega : \mu_1 + \sigma_1 Z = e^{\mu_2 + \sigma_2 Z}\right\}.$$

As $\mathbb{P}(D_3) = 0,\ \mathbb{P}\left(D_1 \bigcup D_2\right) = 1 holds,$ and therefore,

$$d_{FM}(\xi,\eta) = \mathbb{E}\left[(\mu_1 + \sigma_1 Z) - e^{\mu_2 + \sigma_2 Z}\right]\mathbb{I}_{D_1} + \mathbb{E}\left[e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)\right]\mathbb{I}_{D_2}.$$

By definition, the set D_1 is either empty or comprises those ω for which $Z \in (k_1, k_2)$ for some real k_1, k_2 as the graph of a linear function can lie above the graph of an exponent only within a finite interval.

In case of $D_1 = \varnothing$, considering that the expectation of the lognormal distribution with parameters μ_2, σ_2^2 is equal to $\exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right]$, we obtain

$$d_{FM}(\xi,\eta) = E\left[e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)\right] = -\mu_1 + \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right].\#(20)$$

If $D_1 = \left\{\omega : Z \in (k_1, k_2)\right\}$, then as it is much more convenient to work with D_1 than with D_2, we eliminate the indicator $\mathbb{I}_{D_2}$. Using the formula

$$\mathbb{E}X\mathbb{I}_{D_2} = \mathbb{E}X - \mathbb{E}X\mathbb{I}_{D_1}$$

for $X = e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)$, we get

$$d_{FM}(\xi,\eta) = -\mu_1 + \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right] - 2E\left[e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)\right]\mathbb{I}_{D_1}.\#(21)$$

As $\mathbb{P}(D_1) = \Phi(k_2) - \Phi(k_1)$, we need to calculate $\mathbb{E}Z\mathbb{I}_{D_1}$ и $\mathbb{E}e^{\sigma_2 Z}\mathbb{I}_{D_1}$.

To find the first moment of a random variable $Z\mathbb{I}_{D_1}$, we find its Laplace transform

$$\psi(\lambda) = Ee^{-\lambda Z\mathbb{I}_{D_1}} = 1 - P(D_1) + \frac{1}{\sqrt{2\pi}}\int_{k_1}^{k_2}\exp\left[-\lambda x - \frac{x^2}{2}\right]dx =$$
$$= 1 - P(D_1) + \exp\left[\frac{\lambda^2}{2}\right]\left[\Phi(k_2 + \lambda) - \Phi(k_1 + \lambda)\right].$$

As the first moment exists, it is equal to

$$\mathbb{E}Z\mathbb{I}_{D_1} = -\psi'(0) = \phi(k_1) - \phi(k_2).$$

Now, let us find

$$\mathbb{E}e^{\sigma_2 Z}\mathbb{I}_{D_1} = \frac{1}{\sqrt{2\pi}}\int_{k_1}^{k_2}\exp\left[\sigma_2 x - \frac{x^2}{2}\right]dx = \exp\left[\frac{\sigma_2^2}{2}\right]\left[\Phi(k_2 - \sigma_2) - \Phi(k_1 - \sigma_2)\right].$$

Combining the above formulas, we obtain

$$d_{FM}(\xi,\eta) = \mu_1\left(2\left[\Phi(k_2) - \Phi(k_1)\right] - 1\right) + 2\sigma_1\left[\phi(k_1) - \phi(k_2)\right] +$$
$$+ e^{\mu_2 + \frac{\sigma_2^2}{2}}\left(1 - 2\left[\Phi(k_2 - \sigma_2) - \Phi(k_1 - \sigma_2)\right]\right).$$

To obtain the final result, it is necessary to calculate k_1, k_2 and find the conditions under which D_1 is nonempty. If D_1 is nonempty, then k_1, k_2 are the roots of the equation

$$\mu_1 + \sigma_1 x = \exp\left[\mu_2 + \sigma_2 x\right]. \# (22)$$

Now, let us make the variable replacement $y = -\frac{\sigma_2}{\sigma_1}(\mu_1 + \sigma_1 x), x = -\frac{\mu_1}{\sigma_1} - \frac{y}{\sigma_2}$. Then, the equation is transformed into

$$-\frac{\sigma_1}{\sigma_2}y = \exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1} - y\right];$$

$$ye^y = -\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right]. \# (23)$$

The right-hand side is negative, so (23) has two real solutions (i.e., D_1 is nonempty) only in the case of $-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right] > -e^{-1}$ (see the definition of the Lambert W). Taking the logarithm of this inequality, we obtain (*).

If condition (*) is satisfied, the roots of (23) are found using the W function:

$$y_1 = W_0\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right]\right),$$

$$y_2 = W_{-1}\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right]\right).$$

By substituting these solutions into the inverse replacement $x=-\frac{\mu_1}{\sigma_1}-\frac{y}{\sigma_2}$, we obtain (13), which completes the proof of the theorem.

Proof of Corollary 1

If X,Y are random variables, it immediately follows from (6) that

$$d_{FM}\left(cX,cY\right)=\left|c\right|d_{FM}\left(X,Y\right),c\in\mathbb{R}.$$

Thus,
$$d_{FM}\left(X_t^B,X_t^S\right)=X_0 d_{FM}\left(1+\sigma_B W_t,\exp\left[-\frac{\sigma_S^2 t}{2}+\sigma_S W_t\right]\right)=X_0 d_{FM}\left(\xi,\eta\right).$$

Here, we designate $\xi=1+\sigma_B W_t,\eta=\exp\left[-\frac{\sigma_S^2 t}{2}+\sigma_S W_t\right]$. Clearly,

$$\xi\sim N\left(\mu_1,\sigma_1^2\right),\eta\sim LN\left(\mu_2,\sigma_2^2\right),\#(24)$$

where $\mu_1=1,\mu_2=-\frac{\sigma_S^2 t}{2},\sigma_1^2=\sigma_B^2 t,\sigma_2^2=\sigma_S^2 t$.

Let us show that condition (*) is satisfied. Suppose that for some $\sigma_1>0,\sigma_2>0$, this is not true. Then, through (14), $d_{FM}\left(\xi,\eta\right)=0$ (i.e., $\xi\overset{d}{=}\eta$). We obtain the contradiction with (24). Substituting the parameter values into formula (12) of Theorem 1, we obtain (15) and (16).

Proof of Theorem 2

1. Let us fix $\sigma_2>0$ and consider an optimization problem

$$d_{FM}\left(\xi,\eta\right)\to\min_{\sigma_1>0}$$

From (15) and (16) and the continuous differentiability of W for $\sigma_1,\sigma_2>0$, the function $d_{FM}\left(\xi,\eta\right)$ is found to be continuously differentiable with respect to σ_1 at $\sigma_1,\sigma_2>0$. Moreover, the values close to zero and a very large value of σ_1 are not optimal. Hence, the minimum point satisfies the necessary condition

$$\frac{\partial d_{FM}\left(\xi,\eta\right)}{\partial\sigma_1}=0.$$

Substituting into (21) the martingale values of parameters and differentiating it by σ_1 using the Leibniz integral rule, we obtain

$$\frac{\partial d_{FM}\left(\xi,\eta\right)}{\partial\sigma_1}=-2\frac{\partial}{\partial\sigma_1}\mathbb{E}\left[\exp\left[-\frac{\sigma_2^2}{2}+\sigma_2 Z\right]-1-\sigma_1 Z\right]\mathbb{I}_{D_1}=$$
$$=-2\frac{\partial}{\partial\sigma_1}\int_{k_1}^{k_2}\left(\exp\left[-\frac{\sigma_2^2}{2}+\sigma_2 z\right]-1-\sigma_1 z\right)\phi(z)dz=2\int_{k_1}^{k_2}z\phi(z)dz-$$
$$-2\phi(z)\left(\exp\left[-\frac{\sigma_2^2}{2}+\sigma_2 z\right]-1-\sigma_1 z\right)\Big|_{k_1}^{k_2}=2\mathbb{E}Z\mathbb{I}_{D_1}=2\left[\phi(k_1)-\phi(k_2)\right].$$

Here, the term with substitution is equal to zero, as k_1,k_2 are the roots of (22).

Thus, the point σ_1 is optimal if and only if

$$\phi(k_1)=\phi(k_2)\Leftrightarrow\left|k_1\right|=\left|k_2\right|.$$

Let us show that the case $k_1 = k_2$ is impossible. Indeed, if $k_1 = k_2$, then from (15), $d_{FM}(\xi,\eta) = 0$; that is, $\xi \overset{d}{=} \eta$. We obtain the contradiction with

$$\xi \sim \mathcal{N}(\mu_1, \sigma_1^2), \eta \sim \mathcal{LN}(\mu_2, \sigma_2^2).$$

Thus, $k_2 = -k_1$. From (16), we obtain

$$W_0(z) + W_{-1}(z) = -2\delta,$$

where we designate $\delta = \frac{\sigma_2}{\sigma_1}, z = -\frac{\sigma_2}{\sigma_1}\exp\left[-\frac{\sigma_2^2}{2} - \frac{\sigma_2}{\sigma_1}\right]$. Adding to this equation the definition of the Lambert W function, we obtain the system

$$\begin{cases} W_0(z) + W_{-1}(z) = -2\delta \\ W_0(z)e^{W_0(z)} = z \\ W_{-1}(z)e^{W_{-1}(z)} = z \end{cases}$$

Solving it, we determine

$$W_0(z) = -\delta + \sqrt{\delta^2 - (ze^{\delta})^2},$$

$$W_{-1}(z) = -\delta - \sqrt{\delta^2 - (ze^{\delta})^2}.$$

Hence, from (16)

$$k_2 = -k_1 = \frac{1}{\sigma_1}\sqrt{1 - e^{-\sigma_2^2}}.$$

Let us substitute the determined value of k_2 into (22)

$$1 + \sqrt{1 - e^{-\sigma_2^2}} = \exp\left[-\frac{\sigma_2^2}{2} + \frac{\sigma_2}{\sigma_1}\sqrt{1 - e^{-\sigma_2^2}}\right].$$

From this, we can easily express as

$$\sigma_1^* = \frac{\sigma_2\sqrt{1 - e^{-\sigma_2^2}}}{\frac{\sigma_2^2}{2} + \ln\left(1 + \sqrt{1 - e^{-\sigma_2^2}}\right)}.$$

2. Analogically to the first point, we equate to zero the derivative

$$\frac{\partial d_{FM}(\xi,\eta)}{\partial \sigma_2} = -2\frac{\partial}{\partial \sigma_1}\mathbb{E}\left[\exp\left[-\frac{\sigma_2^2}{2} + \sigma_2 Z\right] - 1 - \sigma_1 Z\right]\mathbb{I}_{D_1} =$$

$$= -2\int_{k_1}^{k_2}(z - \sigma_2)\exp\left[-\frac{\sigma_2^2}{2} + \sigma_2 z\right]\phi(z)dz = -2\int_{k_1}^{k_2}(z - \sigma_2)\phi(z - \sigma_2)dz =$$

$$= -2\int_{k_1-\sigma_2}^{k_2-\sigma_2} y\,d\Phi(y) = -2\left[\phi(k_1 - \sigma_2) - \phi(k_2 - \sigma_2)\right] = 0.$$

From here, $|k_1-\sigma_2|=|k_2-\sigma_2|$. Again, considering the impossibility of case $k_1=k_2$, we obtain $k_1+k_2=2\sigma_2$.

Proof of Theorem 3

From (5), we obtain

$$d_{TV}(\xi,\eta)=2\int_A \left(p_\xi(x)-p_\eta(x)\right)dx, \#(25)$$

where set $A=\{x: p_\xi(x)>p_\eta(x)\}$ — is the union of intervals whose endpoints are the roots of the equation

$$p_\xi(x)=p_\eta(x).$$

This equation has only positive roots as $p_\xi(x)>0, p_\eta(x)=0$ at $x\le 0$. Let us write it out explicitly and transform it.

$$\frac{1}{\sigma_1}\exp\left[-\frac{(x-1)^2}{2\sigma_1^2}\right]=\frac{1}{\sigma_2 x}\exp\left[-\frac{(\ln x+\sigma_2^2/2)^2}{2\sigma_2^2}\right];$$

$$\ln x+\ln\left(\frac{\sigma_2}{\sigma_1}\right)=\frac{1}{2\sigma_1^2}\left(x^2-2x+1\right)-\frac{1}{2\sigma_2^2}\left((\ln x)^2+\sigma_2^2\ln x+\frac{\sigma_2^4}{4}\right);$$

$$2\sigma_1^2\ln x+2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right)=\left(x^2-2x\right)+1-\frac{\sigma_1^2}{\sigma_2^2}(\ln x)^2-\sigma_1^2\ln x-\frac{\sigma_1^2\sigma_2^2}{4};$$

$$\left(x^2-2x\right)-\left(\frac{\sigma_1}{\sigma_2}\right)^2(\ln x)^2-3\sigma_1^2\ln x+1-\frac{\sigma_1^2\sigma_2^2}{4}-2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right)=0.$$

Let us denote the left part of the equation by $h(x)$ and find the derivatives of this function:

$$h'(x)=2(x-1)-\left(\frac{\sigma_1}{\sigma_2}\right)^2\frac{2\ln x}{x}-\frac{3\sigma_1^2}{x},$$
$$h''(x)=2-\left(\frac{\sigma_1}{\sigma_2}\right)^2\frac{2(1-\ln x)}{x^2}+\frac{3\sigma_1^2}{x^2}.$$

Equality $h'(x)=0$ is equivalent to $2x(x-1)=2\left(\frac{\sigma_1}{\sigma_2}\right)^2\ln x+3\sigma_1^2$, which has exactly two roots for geometric reasons. Hence, the function $h(x)$ has two local extrema on $(0,+\infty)$. Let us denote them by x_1^*, x_2^* and $x_1^*<x_2^*$.

As $\lim_{x\to 0+}h(x)=-\infty, \lim_{x\to+\infty}h(x)=+\infty$, the equation $h(x)=0$ has $(0,+\infty)$ at most three roots. As $p_\xi(x)>p_\eta(x)$ at $x<0$ and at $x>0$ $p_\xi(x)>p_\eta(x)$, when $h(x)<0$, set A can be represented as

$$A=(-\infty,x_1)\cup(x_2,x_3).\#(26)$$

If the equation has less than three roots, consider $x_2=x_3$. Combining (26) with the integral representation of the total variation metric (25), we obtain the required statement.

To find the Kolmogorov metric, consider the function $g(x)=F_\xi(x)-F_\eta(x)$. As $\lim_{x\to\infty}g(x)=0$ at the point at which the maximum of the modulus is reached, we have the equality $g'(x)=p_\xi(x)-p_\eta(x)=0$.

The solutions of this equation are the roots of x_1,x_2,x_3 obtained in (19). Hence, $d_K(\xi,\eta)=\max_{x\in\mathbb{R}}\left|F_\xi(x)-F_\eta(x)\right|=\max_{i=1,2,3}\left|F_\xi(x_i)-F_\eta(x_i)\right|$.

4 Numerical Analysis

Calculation of the Fortet-Mourier Metric

The value of the Fortet-Mourier metric in (12) cannot be expressed in elementary functions. This is an expected result, which naturally arises when dealing with normal and lognormal distributions: the distribution function $\Phi(\cdot)$ appears, for example, in the Black-Scholes formula (Black and Scholes, 1973). However, in (12) the Lambert W, which is much less frequently used function than $\Phi(\cdot)$. Nevertheless, many mathematical packages allow calculating the value of any of its branches, which simplifies the numerical calculation of the formula.

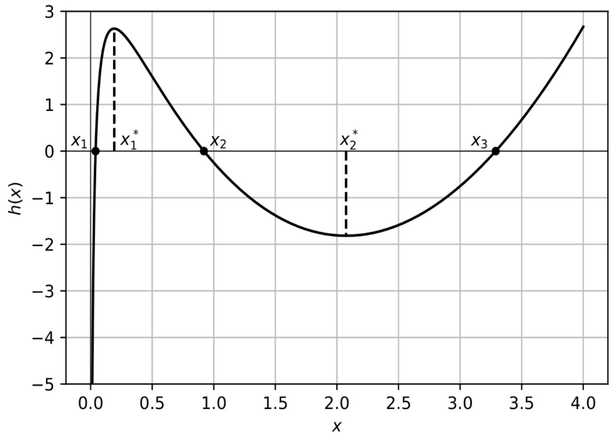

Figure 2. Function graph $h(\cdot)$ at $\sigma_1 = \sigma_2 = 1$

Source: The authors.

Calculation of the Total Variation Metric and the Kolmogorov Metric

Let us discuss here the numerical computation of the total variation metric.

Calculation $d_{TV}(\xi,\eta)$ Using Quadrature Methods

One of the approaches for the calculation of the total variation metric is the calculation (see (5)) of the integral

$$2\int_{\mathbb{R}} (p_\xi(x) - p_\eta(x))^+ dx$$

using quadrature methods.

As

$$(p_\xi(x) - p_\eta(x))^+ \le p_\xi(x),$$

and $\xi \sim \mathcal{N}(\mu_1, \sigma_1)$, we will approximate the integral by the proper one

$$\int_{\mathbb{R}} (p_\xi(x) - p_\eta(x))^+ dx \approx \int_{1-\delta}^{1+\delta} (p_\xi(x) - p_\eta(x))^+ dx.$$

As for $x < 0$,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{t^2}{2}\right] dt \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \frac{t}{x} \exp\left[-\frac{t^2}{2}\right] dt = -\frac{1}{\sqrt{2\pi}x} \exp\left[-\frac{x^2}{2}\right],$$

the approximation error does not exceed

$$2\int_{-\infty}^{1-\delta} p_\xi(x) dx = 2\Phi\left(-\frac{\delta}{\sigma_1}\right) \le \frac{\sigma_1}{\delta}\sqrt{\frac{2}{\pi}} \exp\left[-\frac{\delta^2}{2\sigma_1^2}\right]. \#(27)$$

Now, let us estimate the accuracy of the integral calculation

$$\int_{1-\delta}^{1+\delta} (p_\xi(x) - p_\eta(x))^+ dx$$

by using the trapezoidal method. The integrand function here is not twice continuously differentiable; however, (26) indicates that it has no more than three break points. As the function is zero at each break point, the integration error in the mesh section containing these points does not exceed $3M_1h^2$, where

$$M_1 = \max_{x\in(1-\delta,1+\delta)} \left| p'_\xi(x) - p'_\eta(x) \right|.$$

Combining this with the standard estimation for the trapezoidal rule (Samarsky and Gulin, 1989), we obtain

$$|\Psi| \le \frac{h^2(2\delta)}{12} M_2 + 3M_1h^2,$$

where Ψ is the error incurred in the integration calculations performed on a grid of size N, $h = \frac{2\delta}{N}$ grid step, and $M_2 = \max_{x\in(1-\delta,1+\delta)} \left| p''_\xi(x) - p''_\eta(x) \right|$.

As $p_\xi(x) = \frac{1}{\sigma_1}\phi\left(\frac{x-1}{\sigma_1}\right)$ considering $\phi'(x) = -x\phi(x)$, we find

$$p'_\xi(x) = -\frac{x-1}{\sigma_1^3}\phi\left(\frac{x-1}{\sigma_1}\right), p''_\xi(x) = -\frac{1}{\sigma_1^3}\phi\left(\frac{x-1}{\sigma_1}\right) + \frac{(x-1)^2}{\sigma_1^5}\phi\left(\frac{x-1}{\sigma_1}\right).$$

$$\max_{x\in(1-\delta,1+\delta)} \left| p'_\xi(x) \right| \le \frac{\delta}{\sqrt{2\pi}\sigma_1^3}, \max_{x\in(1-\delta,1+\delta)} \left| p''_\xi(x) \right| \le \frac{1}{\sqrt{2\pi}\sigma_1^3}\left(1 + \frac{\delta^2}{\sigma_1^2}\right).$$

Using $p_\eta(x) = \frac{1}{\sigma_2 x}\phi\left(\frac{\ln x - \mu_2}{\sigma_2}\right)$, we can find

$$p'_\eta(x) = -\frac{\sigma_2 + d}{x^2\sigma_2^2}\phi(d), p''_\eta(x) = \frac{\phi(d(x))}{\sigma_2 x^3}\left(2 + \frac{3d(x)}{\sigma_2} + \frac{(d(x))^3 - 1}{\sigma_2^2}\right),$$

where we designate $d(x) = \frac{\ln x - \mu_2}{\sigma_2}$.

Let us assume that $1-\delta > 0$, which will be true in practice as the values of volatilities are usually small. Let us denote

$$d^* = \max_{x\in(1-\delta,1+\delta)} d(x) = \max\left(\frac{|\ln(1-\delta) - \mu_2|}{\sigma_2}, \frac{|\ln(1+\delta) - \mu_2|}{\sigma_2}\right).$$

Then,

$$\max_{x\in(1-\delta,1+\delta)} \left| p'_\eta(x) \right| \le \frac{\sigma_2 + d^*}{\sqrt{2\pi}\sigma_2^2(1-\delta)^2},$$

$$\max_{x\in(1-\delta,1+\delta)} \left| p''_\eta(x) \right| \le \frac{1}{\sqrt{2\pi}\sigma_2(1-\delta)^3}\left(2 + \frac{3d^*}{\sigma_2} + \frac{(d^*)^3 + 1}{\sigma_2^2}\right).$$

Combining the obtained inequalities, we find

$$|\Psi| \le \frac{2\delta^3}{3\sqrt{2\pi}N^2}\left[\frac{1}{\sigma_1^3}\left(1 + \frac{\delta^2}{\sigma_1^2}\right) + \frac{1}{\sigma_2(1-\delta)^3}\left(2 + \frac{3d^*}{\sigma_2} + \frac{(d^*)^3 + 1}{\sigma_2^2}\right)\right] +$$
$$+ \frac{12\delta^2}{\sqrt{2\pi}N^2}\left[\frac{\delta}{\sigma_1^3} + \frac{\sigma_2 + d^*}{\sigma_2^2(1-\delta)^2}\right].$$

Calculation of $d_{TV}(\xi,\eta)$ ***using the Monte Carlo method***

The same integral can be calculated using the Monte Carlo method, as

$$\int_{\mathbb{R}}(p_\xi(x)-p_\eta(x))^+dx=\int_{\mathbb{R}}(1-\frac{p_\eta(x)}{p_\xi(x)})^+p_\xi(x)dx=\mathbb{E}[1-\frac{p_\eta(\xi)}{p_\xi(\xi)}]^+,$$

where the expectation is taken with respect to the distribution of a random variable $\xi\sim p_\xi(\cdot)$.
We simulate the independent random variables $X_1,\dots,X_n\sim p_\xi(\cdot)$ and approximate the integral by $\frac{1}{n}\sum_{i=1}^{n}Y_i$, where $Y_i=2(1-\frac{p_\eta(X_i)}{p_\xi(X_i)})^+$. The mean-square deviation in this case can be expressed as

$$\mathbb{E}(\frac{1}{n}\sum_{i=1}^{n}Y_i-d_{TV}(\xi,\eta))^2=Var\left[\frac{1}{n}\sum_{i=1}^{n}Y_i\right]\le\frac{2}{n}$$

Numerical Solution of a Nonlinear Equation

Let us now discuss the numerical solution of (19). Consider the case that has exactly three roots (for cases with fewer roots, the algorithm will be similar). As in the proof of Theorem 3,

$$h(x)=\left(x^2-2x\right)-(\frac{\sigma_1}{\sigma_2})^2(\ln x)^2-3\sigma_1^2\ln x+1-\frac{\sigma_1^2\sigma_2^2}{4}-2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right),$$

$$h'(x)=2(x-1)-(\frac{\sigma_1}{\sigma_2})^2\frac{2\ln x}{x}-\frac{3\sigma_1^2}{x},$$

$$h''(x)=2+(\frac{\sigma_1}{\sigma_2})^2\frac{2(\ln x-1)}{x^2}+\frac{3\sigma_1^2}{x^2}.$$

Equation $h''(x)=0$ has exactly one root, which means that the function $h(z)$ has one inflection point that lies between x_1^* и x_2^* (Fig. 2), and therefore, it is concave on $\left(0,x_1^*\right)$ and convex on $\left(x_2^*,+\infty\right)$.

This ensures that Newton's method for the root x_1 with an initial point $x_1^{(0)}$ such that $x_1^{(0)}<x_1$ converges to the root. For the same reason, Newton's method will converge to root x_3 at the initial point $x_3^{(0)}>x_3$. Root x_2 can be localized by the bisection method for $[x_1,x_3]$ and then calculated by Newton's method.

According to Samarsky and Gulin (1989), if $h(\cdot)$ is twice continuously differentiable in the neighborhood $U_r\left(x^*\right)$ of root x^* of the equation $h(x)=0$, and

$$q=\frac{M_2\left|x^*-x^{(0)}\right|}{2m_1}<1, m_1=\inf_{x\in U_r(x^*)}\left|h'(x)\right|, M_2=\sup_{x\in U_r(x^*)}\left|h''(x)\right|.$$

Then, Newton's method converges to x^*, and

$$\left|x^{(k)}-x^*\right|\le q^{2^k-1}\left|x^{(0)}-x^*\right|.\#(28)$$

Thus, for convergence, it is sufficient to assume that in some neighborhood of the root, the second derivative is bounded and the first one is strictly separated from zero.

At $x>x_1^{(0)}$,

$$M_2 \le 2 + 2(\frac{\sigma_1}{\sigma_2})^2 \max\left(\frac{\left(1-\ln x_1^{(0)}\right)}{(x_1^{(0)})^2}, 1\right) + \frac{3\sigma_1^2}{(x_1^0)^2},$$

and at the localization of the root, the minimum of the modulus of the first derivative is attained at one of the segment endpoints, where it can be computed explicitly. Therefore, by partitioning the segment until $q<1$, we can achieve a guaranteed rate of convergence (28).

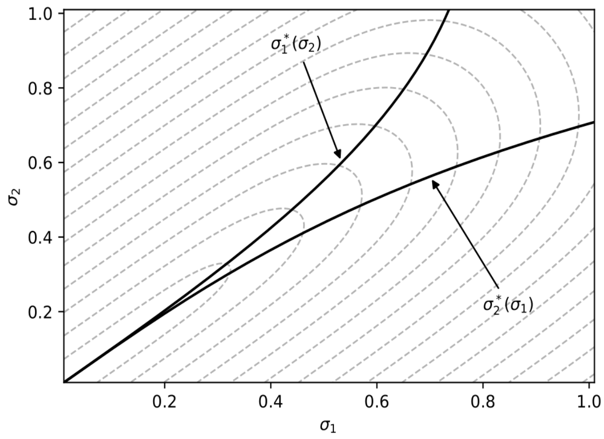

Figure 3. Contour lines $d_{FM}(\xi,\eta)$ and optimum values $\sigma_1^*(\sigma_2), \sigma_2^*(\sigma_1)$ plots

Source: The authors.

Results of Numerical Calculations

The results of metric calculation and optimal values σ_1^*, σ_2^* for the Fortet-Mourier metric at $\sigma_1, \sigma_2 \in (0,1), \mu_1 = 1, \mu_2 = -\frac{\sigma_2^2}{2}$ are presented in Figs. 3 and 4.

The contour lines show that the distances between random variables ξ, η tend to zero as $\sigma_1 \to 0, \sigma_2 \to 0$. This is because of the convergence of distributions ξ, η to the Dirac measure as the volatilities tend to zero.

Application of the Estimates to Certain Options

In this section and hereafter, when referring to processes (1) and (2), we imply that they are martingales; that is, (3) is satisfied.

Estimates (7)–(9), as well as the formulas for the metrics, show that the significant parameters determining the difference between the models are the integrated (or cumulative) volatilities, denoted by σ_1, σ_2.

The application of estimates (7)–(9) to some types of options is shown below.

Put and Call Options

The payoff function of a standard call option $f_C(X_T) = (X_T - K)^+$ is Lipschitz continuous with the Lipschitz constant equal to 1. Therefore, from (7),

$$\left|P_B(f,T) - P_S(f,T)\right| \le d_{FM}\left(X_T^B, X_T^S\right) = X_0 d_{FM}(\xi,\eta).$$

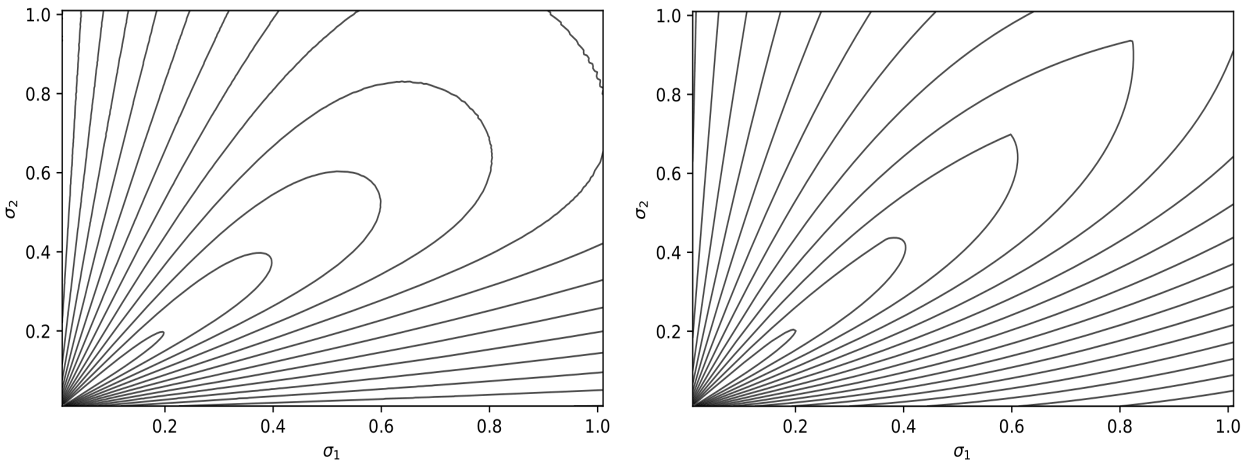

Figure 4. Contour lines $d_{TV}(\xi,\eta)$ and $d_K(\xi,\eta)$

Source: The authors.

Let us use the data obtained by Bachelier (1900). Consider an option with the time to exercise equal to one month, for which the integral volatility equals $\sigma = \sigma_1 = \sigma_2 \approx 0.008$. Then, we find

$$\left|P_B\left(f_C, T\right) - P_S\left(f_C, T\right)\right| \leq 3.1 \cdot 10^{-5} X_0 . \#(29)$$

Exactly the same estimate is true for a put option.

It is also interesting to compare this estimate with that obtained by Schachermayer and Teichmann (2005) for a call option "at the money" (i.e., for $K = X_0$):

$$0 \leq P_B\left(f_C, T\right) - P_S\left(f_C, T\right) \leq \frac{X_0 \sigma^3}{12\sqrt{2\pi}}.$$

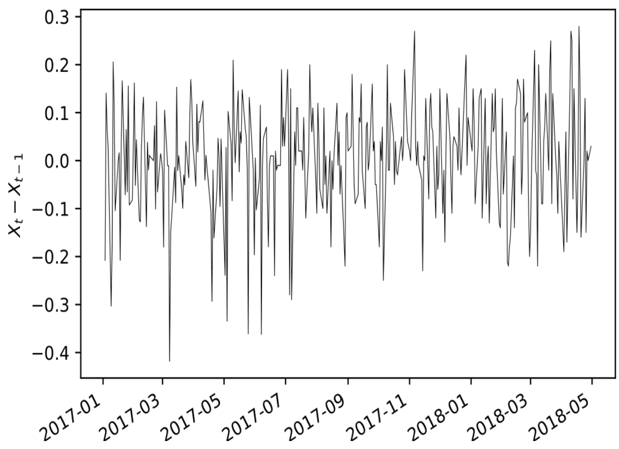

Figure 5. Daily price increments

Source: The authors.

For the same value of σ on the right-hand side, we get $\approx 1.6 \cdot 10^{-8} X_0$. Of course, this exceeds the accuracy of (29) by three orders of magnitude; however, the estimation with the Fortet–Mourier metric allows us to work with a very wide class of payoff functions and therefore is a more universal method.

Binary Options

Consider a binary call option with payout function

$$f_{B,C}\left(X_T\right) = M \mathbb{I}_{X_T \geq K}.$$

Then, from (8),

$$\left|P_B\left(f_{B,C}, T\right) - P_S\left(f_{B,C}, T\right)\right| \leq M d_{TV}\left(X_T^B, X_T^S\right).$$

Substituting the Bachelier's data, we obtain

$$\left|P_B\left(f_{B,C}, T\right) - P_B\left(f_{B,C}, T\right)\right| \leq 6 \cdot 10^{-3} M.$$

As it was noted, the total variation metric provides less accurate but still acceptable estimate.

Let us also apply (9):

$$\left|P_B\left(f_{B,C}, T\right) - P_B\left(f_{B,C}, T\right)\right| \leq M d_K\left(X_T^B, X_T^S\right) \approx 1.6 \cdot 10^{-3} M.$$

The Kolmogorov metric gives a more accurate result, which, however, has the same order as that of the total variation metric.

Estimation of Volatility Using the Oil Market Prices

Let us now try to apply the obtained estimates to the current data. For this purpose, it is necessary to evaluate the parameters σ_B, σ_S of models (1) and (2). Furthermore, we apply statistical estimation methods assuming that the data satisfy the Bachelier model or the Samuelson model. For real market prices, the distribution of their increments or the increments of their logarithms is poorly approximated by the normal distribution and the increments themselves are not independent (e.g., the effect of volatility clusters occurs). These effects are considered using time-series models with conditional heterogeneity (ARCH models) that allow to describe the asset price behavior more precisely. In addition, the processes obtained using these models, with appropriate normalization, converge to diffusion ones (Gouriéroux, 1997; Th. 5.15).

It can justify their application to the estimation of parameters of Bachelier and Samuelson models. However, when comparing these models, we are interested in a rough evaluation of the volatility[9].

Consider price X_t as the closing price for WisdomTree WTI Crude Oil from January 2017 to November 2018 (Figure 5). Let us consider dimensionless values

$$Y_t = \frac{X_t}{X_0}, t = 0,1,\ldots,n = 335.$$

According to the Bachelier model, the price increments $\Delta Y_t = Y_t - Y_{t-1}$ can be represented as

$$\Delta x_t = \alpha + \sigma_B \Delta W_t, \Delta W_t = W_t - W_{t-1} \sim \mathcal{N}(0,1).$$

Thus, as the Wiener process increments are independent, we consider $\{\Delta x_t\}$ as a sample of random variables having a normal distribution $\mathcal{N}(\alpha, \sigma_B^2)$.

The maximum likelihood estimate $\hat{\sigma}_B$ for the standard deviation from the sample obtained from the Gaussian distribution with two unknown parameters, mathematical expectation and variance, is

$$\hat{\sigma}_B = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(\Delta Y_t - \overline{\Delta Y_t})^2},$$

where

$$\overline{\Delta Y_t} = \frac{1}{n}\sum_{t=1}^{n}\Delta Y_t.$$

This estimate gives an approximate value for the volatility $\hat{\sigma}_B \approx 0.0144$.

In the Samuelson model, the logarithm increments

$$\Delta(\ln Y_t) = \gamma + \sigma_S \Delta W_t \sim \mathcal{N}(\gamma, \sigma_S^2).$$

Estimating the standard deviation similarly, we obtain $\hat{\sigma}_S \approx 0.0150$.

Let us construct a confidence interval for the obtained estimates with confidence level q. For the sample $Z_1,\ldots,Z_n$ obtained from normal distribution with two unknown parameters, the mathematical expectation μ and variance σ^2, the random variable $\sum_{i=1}^{n}\frac{(Z_i - \overline{Z}_n)^2}{\sigma^2}$ has a distribution of $\chi^2(n-1)$ (e.g., DeGroot and Schervish (2011)). Therefore, to estimate the maximum likelihood $\acute{\sigma}$ of the scale parameter σ, we have

$$\mathbb{P}\left(\gamma_1 < n\frac{\hat{\sigma}^2}{\sigma^2} < \gamma_2\right) = \chi_{n-1}(\gamma_2) - \chi_{n-1}(\gamma_1) = q,$$

where $\chi_{n-1}(\cdot)$ denotes the cumulative distribution function for the law $\chi^2(n-1)$. Let us choose

$$\gamma_1 = \chi_{n-1}^{-1}\left(\frac{1-q}{2}\right), \gamma_1 = \chi_{n-1}^{-1}\left(\frac{1+q}{2}\right),$$

then the corresponding confidence interval for σ is

$$\left[\hat{\sigma}\sqrt{\frac{n}{\gamma_2}},\hat{\sigma}\sqrt{\frac{n}{\gamma_1}}\right].$$

For the confidence level $q = 0.99$, we obtain the confidence intervals as follows:

$$\sigma_B \in [0.0131, 0.0160], \sigma_S \in [0.0136, 0.0166]. \# (30)$$

The obtained results are consistent with the normalized values of Chicago Board Options Exchange (CBOE) Oil Volatility Index (OVX) over the same period of time (Fig. 6). This index is calculated similarly to the volatility index (VIX) but uses oil options. The OVX values should be interpreted as implicit volatility (i.e., volatility calculated based on the observed option prices and reflecting appropriate expectations of market volatility behaviour in the next month). By contrast, the estimates derived from the historical data $\hat{\sigma}_B, \hat{\sigma}_S$ reflect the value of realized volatility; therefore, the comparison of these values is not entirely correct. Nevertheless, our goal is to only estimate the order of magnitudes σ_B и σ_S; thus, it is acceptable for a rough evaluation of "engineering character."

Now we apply the estimate (7) to the call option with the time to expiration equal to one month ($T = 30$) and obtain

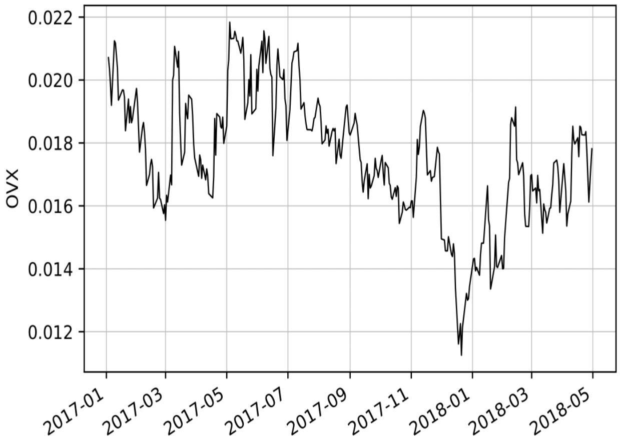

Figure 6. OVX index

Source: The authors.

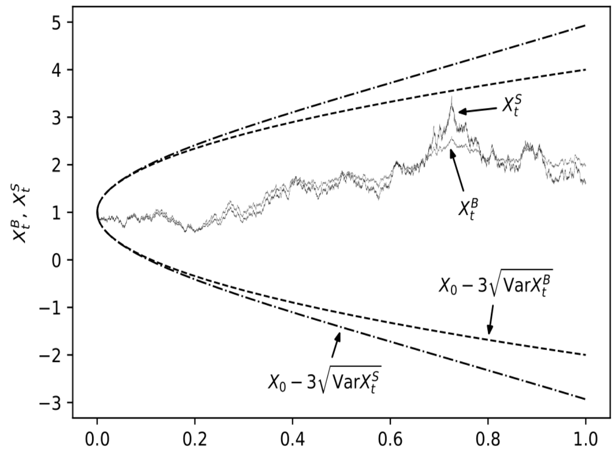

Figure 7. Process trajectories X_t^B, X_t^S.

Source: The authors.

$$\left|P_B(f_C,T) - P_S(f_C,T)\right| \le d_{FM}\left(X_T^B, X_T^S\right) \approx 4.7 \cdot 10^{-3} X_0. \# (31)$$

For a binary option with $T = 30$ and payout M, according to (8),

$$\left|P_B(f_B,T) - P_S(f_B,T)\right| \le M d_{TV}\left(X_T^B, X_T^S\right) \approx 7.9 \cdot 10^{-2} M. \# (32)$$

If we apply (9), we obtain

$$\left|P_B(f_B,T) - P_S(f_B,T)\right| \le M d_K\left(X_T^B, X_T^S\right) \approx 2.1 \cdot 10^{-2} M. \# (33)$$

Values of integral volatility

Let us find at what values of the integral volatility parameter the processes X_t^B, X_t^S remain "close" to each other.

Using the Ito formula (e.g., Øksendal, 1991), we find that X_t^B, X_t^S satisfy the stochastic differential equations

$$dX_t^B = \sigma_B X_0 dW_t,$$

$$dX_t^S = \sigma_S X_t^S dW_t,$$

where for a small t value, the optimal relation between the volatilities is $\sigma_B \approx \sigma_S$.

Let us now calculate the variances:

$$VarX_t^B = X_0^2\sigma_B^2 t, \sqrt{VarX_t^B} = X_0\sigma_B\sqrt{t},$$

$$VarX_t^S = X_0^2\left(e^{\sigma_S^2 t}-1\right), \sqrt{VarX_t^S} = X_0\sqrt{\left(e^{\sigma_S^2 t}-1\right)}.$$

The variances and standard deviations depend only on the initial price and integral volatility. Assuming $X_0 = 1, \sigma = \sigma_B = \sigma_S = 1$, let us model both processes (Fig. 7) such that they correspond to the same Wiener process W_t. At $t \approx 0.2$, the standard deviations and the processes themselves begin to differ appreciably. This value corresponds to the integral volatility value $\sigma\sqrt{t} \approx 0.45$.

For the options considered in the previous section, the integral volatility is approximately equal to $\hat{\sigma}\sqrt{T} \approx 0.015 \cdot \sqrt{30} \approx 0.082$.

Option Price Sensitivity to Volatility

To validate the above-used estimates (31)–(33), the option price must change insignificantly for small changes in volatility. This requirement is based on the fact that the value σ is never exactly known in the model and its estimation leads to an error when calculating the option price. Let us estimate the sensitivity vega (see Hull, 2012)

$$\mathcal{V} = \frac{\partial P(f,T)}{\partial \sigma}$$

for standard and binary put and call options.

The price of a standard call option in the Bachelier model is calculated as

$$P_B(f_C,T) = (X_0 - K)\Phi\left(\frac{X_0 - K}{\sigma_B\sqrt{T}X_0}\right) + \sigma_B\sqrt{T}X_0\phi\left(\frac{X_0 - K}{\sigma_B\sqrt{T}X_0}\right).$$

Its derivation has been provided by Schachermayer and Teichmann (2005). Similarly, the price of a standard put option can be determined:

$$P_B(f_P,T) = (K - X_0)\Phi\left(\frac{K - X_0}{\sigma_B\sqrt{T}X_0}\right) + \sigma_B\sqrt{T}X_0\phi\left(\frac{K - X_0}{\sigma_B\sqrt{T}X_0}\right).$$

Let us find the vega coefficient for these options:

$$\frac{\partial P_B(f_C,T)}{\partial \sigma_B} = (X_0 - K)\phi\left(\frac{X_0 - K}{\sigma_B\sqrt{T}X_0}\right)\left(-\frac{X_0 - K}{\sigma_B^2\sqrt{T}X_0}\right) + \sqrt{T}X_0\phi\left(\frac{X_0 - K}{\sigma_B\sqrt{T}X_0}\right) +$$

$$+\sigma_B\sqrt{T}X_0\phi'\left(\frac{X_0 - K}{\sigma_B\sqrt{T}X_0}\right)\left(-\frac{X_0 - K}{\sigma_B^2\sqrt{T}X_0}\right) = X_0\sqrt{T}\phi\left(\frac{X_0 - K}{\sigma_B\sqrt{T}X_0}\right),$$

as $\phi'(x) = -x\phi(x)$.

Similarly, for a put option,

$$\frac{\partial P_B(f_P,T)}{\partial \sigma_B} = X_0\sqrt{T}\phi\left(\frac{K - X_0}{\sigma_B\sqrt{T}}\right) = \frac{\partial P_B(f_C,T)}{\partial \sigma_B}. \#(34)$$

In the Samuelson model, the prices of standard put and call options are determined using the Black-Scholes formulas:

$$P_S\left(f_C,T\right)=X_0\Phi\left(\frac{\ln\frac{X_0}{K}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)-K\Phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right),$$

$$P_S\left(f_P,T\right)=-X_0\left[1-\Phi\left(\frac{\ln\frac{X_0}{K}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)\right]+K\left[1-\Phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)\right].$$

The derivatives of these quantities obtained by σ_S are found to coincide. Denoting

$$y_+=\frac{\ln\frac{X_0}{K}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}},\; y_-=\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}},$$

let us find

$$\frac{\partial P_S\left(f_C,T\right)}{\partial\sigma_S}=\frac{\partial P_S\left(f_P,T\right)}{\partial\sigma_S}=X_0\phi\left(y_+\right)\left(-\frac{\ln\frac{X_0}{K}}{\sigma_S^2\sqrt{T}}+\frac{1}{2}\sqrt{T}\right)-K\phi\left(y_-\right)\left(-\frac{\ln\frac{X_0}{K}}{\sigma_S^2\sqrt{T}}-\frac{1}{2}\sqrt{T}\right).\#(35)$$

For binary call and put options with payout features,

$$f_{B,C}\left(X_T\right)=M\mathbb{I}_{X_T>K},f_{B,P}\left(X_T\right)=M\mathbb{I}_{X_T<K}.$$

Accordingly, the price is determined as an expectation with respect to the martingale measure:

$$P_B\left(f_{B,C}\right)=\mathbb{E}^Bf_{B,C}\left(X_T\right)=M\mathbb{P}^B\left(X_T>K\right)=M\left(1-\Phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right)\right),$$

$$P_B\left(f_{B,P}\right)=M\Phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right),$$

$$P_S\left(f_{B,C}\right)=\mathbb{E}^Sf_{B,C}\left(X_T\right)=M\mathbb{P}^S\left(X_T>K\right)=M\Phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right),$$

$$P_S\left(f_{B,P}\right)=M\Phi\left(\frac{\ln\frac{K}{X_0}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right).$$

From this, we find

$$\frac{\partial P_B\left(f_{B,C},T\right)}{\partial\sigma_B}=-\frac{\partial P_B\left(f_{B,P},T\right)}{\partial\sigma_B}=M\phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right)\left(\frac{K-X_0}{\sigma_B^2\sqrt{T}X_0}\right),\#(36)$$

$$\frac{\partial P_S\left(f_{B,C},T\right)}{\partial\sigma_S}=-\frac{\partial P_S\left(f_{B,P},T\right)}{\partial\sigma_S}=M\phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)\left(-\frac{\ln\frac{X_0}{K}}{\sigma_S^2\sqrt{T}}-\frac{1}{2}\sqrt{T}\right).\#(37)$$

Let us now estimate the order of the price calculation error that appears due to an inaccurate measure of volatility. This error approximately equals to $|\mathcal{V}\Delta\sigma|$, where $\mathcal{V}$ is the option vega coefficient and $\Delta\sigma$ is the volatility measurement error. As options «at the money» have the greatest liquidity, their study is of the greatest interest. Therefore, we further assume that $K = X_0, T = 30$. From (34) and (35) considering confidence intervals (30), we obtain that for the standard options with the confidence probability, equal to 0.99, the error approximation of $\left|P_B\left(f_C,T\right)-P_S\left(f_C,T\right)\right|$ calculation does not exceed

$$\sqrt{\frac{T}{2\pi}}X_0\max\left|\Delta\sigma_B\right|+\sqrt{T}\phi\left(\frac{1}{2}\hat{\sigma}_S\sqrt{T}\right)X_0\max\left|\Delta\sigma_S\right|\approx 7\cdot 10^{-3}X_0$$

For binary options with $K = X_0, T = 30$, according to (36) and (37), with confidence probability 0.99, the error approximation does not exceed

$$\frac{1}{2}M\sqrt{T}\phi\left(\frac{1}{2}\hat{\sigma}_S\sqrt{T}\right)\max\left|\Delta\sigma_S\right|\approx 1.8\cdot 10^{-3}M.$$

The resulting estimates differ from (31)-(33) by no more than an order of magnitude. Thus, with the estimation methods used, the error associated with an inaccurate measurement of the volatility can make almost the same contribution to the option price as a model change.

In this section, sensitivity estimation is obtained only for the options of a special form. When applying similar methods for classes of functions, the accuracy of the estimation deteriorates considerably. Let us estimate the vega coefficient in the Bachelier model: if we denote $p(\cdot)$ as the density of the random variable $\frac{X_T}{X_0}$, then the price of the European option with payout function $f(\cdot)$ and time to expiration T can be found as $P_B\left(f,T\right)=\int_{-\infty}^{\infty}f\left(yX_0\right)p\left(y\right)dy.$

Based on (1) and (3), the function $p(\cdot)$ can be expressed as $p\left(y\right)=\frac{1}{\sigma_B\sqrt{T}}\phi\left(\frac{y-1}{\sigma_B\sqrt{T}}\right).$

After changing the variables $z=\frac{y-1}{\sqrt{T}}$, we obtain

$$P_B\left(f,T\right)=\int_{-\infty}^{\infty}f\left(\left(1+\sqrt{T}z\right)X_0\right)\frac{1}{\sigma_B}\phi\left(\frac{z}{\sigma_B}\right)dz.$$

Let us differentiate the integral by parameter σ_B. The differentiation performed under the integral is possible for all $\sigma_B > 0$, as, considering σ_B on each finite interval, the function $\left(f\frac{\partial p}{\partial\sigma_B}\right)(\cdot)$ will be majorized by an integrable function that does not depend on σ_B.

$$\frac{\partial P_B}{\partial\sigma_B}\left(f,T\right)=\int_{-\infty}^{\infty}f\left(\left(1+\sqrt{T}z\right)X_0\right)\left[-\frac{1}{\sigma_B^2}\phi\left(\frac{z}{\sigma_B}\right)+\frac{z^2}{\sigma_B^4}\phi\left(\frac{z}{\sigma_B}\right)\right]dz=$$
$$=\frac{1}{\sigma_B}\int_{-\infty}^{\infty}f\left(\left(1+\sqrt{T}\sigma_B z\right)X_0\right)\left[-\phi\left(z\right)+z^2\phi\left(z\right)\right]dz. \#(38)$$

For a bounded function $f(\cdot)\in B(\mathbb{R})$,

$$\left|\frac{\partial P_B}{\partial\sigma_B}\left(f,T\right)\right|\le\frac{\|f\|_B}{\sigma_B}\int_{-\infty}^{\infty}\left[\phi\left(z\right)+z^2\phi\left(z\right)\right]dz=\frac{2}{\sigma_B}\|f\|_B.\#(39)$$

For the Lipschitz continuous functions, we will use the inequality

$$\left|f(x)\right| \le \left|f(X_0)\right| + \| f \|_{Lip} \left|x - X_0\right|.$$

Considering that

$$\int_{-\infty}^{\infty} |x|\,\phi(x)dx = \frac{2}{\sqrt{2\pi}}, \int_{-\infty}^{\infty} |x|^3\,\phi(x)dx = \frac{4}{\sqrt{2\pi}},$$

$$\begin{aligned}\left|\frac{\partial P_B}{\partial \sigma_B}(f,T)\right| &\le \frac{1}{\sigma_B}\int_{-\infty}^{\infty}\left(\left|f(X_0)\right| + \| f \|_{Lip}\sqrt{T}\sigma_B |z| X_0\right)\left[\phi(z) + z^2\phi(z)\right]dz \le \\ &\le \frac{2\left|f(X_0)\right|}{\sigma_B} + \frac{6}{\sqrt{2\pi}}\sqrt{T}X_0 \| f \|_{Lip}\end{aligned} \quad \#(40)$$

According to estimates (39) and (40), as well as the confidence interval (30), the calculation error $P_B(f,T)$ for a standard call (put) option in money with $T = 30$ does not exceed

$$\frac{6}{\sqrt{2\pi}}\sqrt{30}\max\left|\Delta\sigma_B\right| X_0 \approx 2\cdot 10^{-2}\cdot X_0,$$

and for a binary option with $K = X_0, T = 30$ does not exceed

$$\frac{2}{\hat{\sigma}_B} M = 0.22\cdot M.$$

The resulting accuracy estimates are inferior to those obtained using the exact representation of the vega coefficient for these options by one or two orders of magnitude, which is expected as a consequence of the universality of the estimates.

5 Conclusion

The approach based on the use of probability metrics enables the estimation of how much the transition from one model to another affects the price of a European option with a payout function from a certain class (represented as a sum of Lipschitz continuous and bounded functions). This price change can be estimated by using an appropriate probabilistic metric and the norm (or semi-norm) of the payout function in a suitable function space. However, the main factor affecting the value of the estimation is the integral volatility, at a large value of which the Bachelier and Samuelson models, which are essentially arithmetic and geometric random walks, cease to be similar. As expected, the estimates obtained using the Fortet-Mourier metric were the most accurate, whereas the use of the total variation metric and the Kolmogorov metric led to similarly less accurate results. Moreover, the calculation of the latter two metrics was reduced to the numerical solution of the same nonlinear equation describing the points of intersection of normal and lognormal densities.

For the oil market, measures of realized volatility were estimated and confidence intervals were constructed assuming that the models are true. By calculating the sensitivity (vega coefficient) for standard and binary options, the error arising in the estimation of model parameters was found to be comparable to the change in price when the model changed.

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Footnotes

[1] Apparently, Samuelson was the first economist to propose this modification of the Bachelier model. Therefore, we use the term "Samuelson's model."

[2] For a complete list of contracts, see CME Group Advisory Notice 20–171, 2020.

[3] This follows directly from the Ito formula.

[4] The assumptions made in Bachelier's thesis (in an informal way) actually mean that the price process is a martingale.

[5] The term "coupling" is also used in random process theory in a different sense; see, for example, Sverchkov, and Smirnov (1990).

[6] The generalized inverse function defined in this manner is also left-continuous. In this case, the random variable $F^{-1}(U)$, where U is uniformly distributed on $(0,1)$ random variable, has a distribution function equal to F.

[7] This metric forms the basis of the nonparametric criterion of the same name, which is based on the theorem proved by Kolmogorov (1933).
[8] Also, Kantorovich metric, Wasserstein metric, and Dudley metric. The variety of names can be explained by many equivalent representations (for details, see Rüschendorf, https://wwwhttps://www.encyclopediaofmath.org/index/index.php?title=Wasserstein_metric=Wasserstein_metric).
[9] An exposition of the statistical analysis concerning volatility has been presented by Melnikov, Volkov, and Nechaev (2001), paragraph 4.3. In contrast to this study, we use the maximum likelihood estimation (instead of an unbiased estimation with uniformly minimal variance) for the volatility, as such estimation for bijective transformation of the parameter reduces to this transformation of the parameter estimate. Among other things, this is applicable when determining implicit volatility.

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