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An Interest Rate Model for Uncertain-Stochastic Financial Markets

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ABSTRACT

Over the past decades, financial markets have increasingly exhibited features of both randomness and uncertainty, creating challenges for interest rate models that rely solely on stochastic or uncertain processes. These models often fail to adequately capture the dual nature of indeterminacy, limiting their relevance in volatile and unpredictable market conditions. This study **aims** to design and assess an interest rate model for uncertain-stochastic financial markets and to derive a framework for zero-coupon bond pricing under this setting. The **methodology** applies uncertain stochastic differential equations, which integrate elements of both probability theory and uncertainty theory, thereby accommodating aleatory and epistemic forms of indeterminacy. The proposed model extends the classical short-rate frameworks by introducing two sources of indeterminacy and provides theoretical derivations for bond pricing. Numerical illustrations are included to demonstrate the application of the model to zero-coupon bond valuation and to highlight differences from conventional approaches. The **findings** indicate that interest rates and zero-coupon bond prices in uncertain stochastic financial markets can be effectively modeled through uncertain random processes, leading to improved pricing accuracy and risk management in environments characterised by incomplete information and unpredictable shocks. The key **conclusion** is that incorporating uncertain stochastic differential equations into the interest rate and zero-coupon bonds' prices modelling offers a more robust and flexible framework for uncertain stochastic markets. This study contributes to the growing body of uncertain stochastic finance by underscoring the need for hybrid models capable of guiding policymakers, investors and financial institutions in ensuring stability and resilience under future market uncertainties.

Keywords: indeterminacy; randomness; uncertainty; interest rate model; zero-coupon bond pricing; probability theory; uncertainty theory; chance theory; uncertain stochastic financial markets; uncertain random processes; uncertain stochastic differential equations

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ОРИГИНАЛЬНАЯ СТАТЬЯ

Модель процентных ставок для неопределенно-стохастических финансовых рынков

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АННОТАЦИЯ

За последние десятилетия финансовые рынки все чаще демонстрируют черты как случайности, так и неопределенности, что создает трудности для моделей процентных ставок, основанных исключительно на стохастических или неопределенных процессах. Эти модели часто неадекватно отражают двойственную природу неопределенности, что ограничивает их применимость в волатильных и непредсказуемых рыночных условиях. **Целью** данного исследования является разработка и оценка модели процентной ставки для неопределенно-стохастических финансовых рынков и разработка модели ценообразования облигаций с нулевым купоном в этих условиях. **Методология** использует неопределенные стохастические дифференциальные уравнения, которые объединяют элементы как теории вероятностей, так и теории неопределенности, тем самым учитывая алеаторные и эпистемические формы неопределенности. Предлагаемая модель расширяет классические модели краткосрочных ставок, вводя два источника неопределенности и предоставляя теоретические выводы для определения цены облигаций. Приведены числовые иллюстрации для демонстрации применения модели к оценке облигаций с нулевым купоном и для выявления отличий от традиционных подходов. **Результаты** исследования показывают, что процентные ставки и цены облигаций с нулевым купоном на неопределенных стохастических финансовых рынках могут быть эффективно смоделированы с помощью неопределенных случайных процессов, что приводит к повышению точности ценообразования и управлению рисками в условиях неполной информации и непредсказуемых шоков. **Ключевой вывод** заключается в том, что включение неопределенных стохастических дифференциальных уравнений в моделирование процентных ставок и цен облигаций с нулевым купоном обеспечивает более надежную и гибкую структуру для неопределенных стохастических рынков. Данное исследование вносит **вклад** в растущий объем знаний неопределенных стохастических финансов, подчеркивая необходимость гибридных моделей, способных помочь политикам, инвесторам и финансовым учреждениям обеспечить стабильность и устойчивость в условиях будущей рыночной неопределенности.

Ключевые слова: неопределенность; случайность; неточность; модель процентной ставки; ценообразование облигаций с нулевым купоном; теория вероятностей; теория неопределенности; теория случайности; неопределенные стохастические финансовые рынки; неопределенные случайные процессы; неопределенные стохастические дифференциальные уравнения

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Introduction

Financial decisions, in practice, are executed under the condition of indeterminacy. Uncertainty and randomness are two common kinds of indeterminacy [1, 2]. Probability theory, introduced by [3], deals with randomness and uncertainty theory, developed by [4] and enhanced by [5], models human subjective uncertainty. Matenda and Chikodza postulated that the probability theory is implemented when the sample size is large to generate the probability distribution from the existing frequency [6]. On the contrary, if the size of the sample is nonexistent or too small to generate the probability distribution, the theory of uncertainty is implemented [6]. In this case, domain specialists are requested to assess their belief degrees of each event occurring [7–9]. Implementing probability theory in this situation results in counterintuitive results. Using uncertainty theory ensures that no counterintuitive results arise [10].

Stochastic processes, random variables and stochastic differential equations (SDEs) are essential in probability theory because they are implemented to deal with random phenomena that change with time [11, 12]. The Brownian motion is one of the broadly implemented stochastic processes in practice [13, 14]. SDEs are powered by stochastic processes. Applying probability theory in the finance discipline resulted in the birth of the theory of stochastic finance. Hence, stochastic processes and SDEs are essential tools in stochastic financial markets. Since the publication of the classical work of [15], SDEs have been extensively implemented in finance theory. The work [15] propounded that the price of a stock can be explained by an exponential Brownian motion and then designed option pricing formulae for the European options.

One of the most important mathematical frameworks in finance is the short-rate interest model, which describes the progression of interest rates (IRs) over time. This framework focuses on the short-term interest rate (IR), which we can simply call the short rate. It is applicable for the shortest period and is often interpreted as an instantaneous rate. Short-rate interest models have been widely used in the IR derivatives pricing, bond valuation and risk management. Traditional short rate models make use of SDEs to elucidate short-term IR progression. Quite a number of models incorporate mean reversion, which is the tendency of IRs moving towards, over time, a long-run average. These short rate models are usually developed in the context of the risk-neutral measure framework.

In 1973, [16] explained the IR by implementing stochastic processes to establish the zero-coupon bond price. The most common stochastic short rate models have been developed by [7–10] and references thereof. A general form of the IRs term structure was examined by [17]. The author suggested a novel mean-reverting IR model powered by a Wiener process. In 1986, [18] developed [16]'s IR model, assuming a no-arbitrage principle. For more expositions on the implementation of SDEs in IR modelling, see, for instance, [19, 20]. Fundamentally, stochastic financial models are premised on the supposition that asset prices are only subject to random movements [21, 22].

Some short-rate models are computationally efficient, practical for real-world applications, incorporate mean reversion, are consistent with observed market prices, and are used in a wide range of applications that are easy to understand and apply. However, some of them allow for negative IRs, assume a single source of uncertainty, have calibration challenges, are unable to fit the entire yield curve, are over-simplified, assume constant volatility, are difficult when pricing complex derivatives and are computationally expensive.

In uncertainty theory, uncertain processes, uncertain variables, and uncertain differential equations (UDEs) are essential because they explain dynamic uncertain systems [23, 24]. The Liu process [5] is a commonly implemented uncertain process. UDEs are driven by uncertain processes. The application of uncertainty theory in the discipline of finance resulted in the emergence of the theory of uncertain finance. As a result, uncertain processes and UDEs are essential tools in stochastic financial markets. UDEs were first applied in financial models by [5]. [5] postulated that the price of a stock could be explained by an exponential Liu process. The author [5] further priced the European options for stocks premised on an uncertain stock model. Since the publication of the classical work of [5], UDEs have been widely adopted in finance theory (see, for instance, [25–27]).

UDEs have been broadly implemented to model rates of interest in uncertain financial markets [28–30]. In an uncertain environment, [31] presumed that the rate of interest is an uncertain process and applied UDEs to describe the IR and priced, in analytic form, a zero-coupon bond. [31] designed the initial uncertain IR model for uncertain markets, even though the rate of interest may be negative in this model. [32] developed the pricing formulae for IR floors and ceilings. Implementing an uncertain fractional differential equation, [33] designed an IR model. [34] generated an IR model using an uncertain exponential Ornstein-Uhlenbeck equation. Some authors, specifically [35, 36], demonstrated that UDEs could model IRs.

Most studies have modelled indeterminacy by independently using randomness or uncertainty. However, [37–40] revealed that uncertainty and randomness could concurrently materialize in a process. Similarly, [21] indicated that, in reality, financial markets frequently comprise human uncertain factors and stochastic factors. Randomness is the aleatory uncertainty, while Liu's uncertainty is the epistemic type. Recent developments in various fields have shown that it is essential to include both randomness and uncertainty when modelling indeterminacy. Hence, studies on indeterminacy have led to novel discoveries on how to describe processes with both uncertainty and randomness. [37] introduced the chance theory to deal with both randomness and uncertainty in sophisticated mathematical systems. Interestingly, [6] propounded that probability theory and uncertainty theory supplement one another.

A chance measure, a chance space, a chance distribution, an uncertain random variable, expected value, and variance were introduced by [37]. In 2015, [41] presented an uncertain random process to describe the uncertain random phenomena dynamics that change with time. Uncertain stochastic differential equations and uncertain random processes are central to the discipline of uncertain random calculus because they describe the evolution of different processes with randomness and uncertainty. The USDE is powered by a canonical Liu process and a Brownian motion [42]. USDEs are driven by US processes. The adoption of chance theory in the discipline of finance resulted in the establishment of the US finance theory. Hence, uncertain stochastic (US) processes and USDEs are imperative tools in US financial markets.

B. Liu [43] examined several aspects of uncertain random variables, which include problems in mathematical programming, risk analysis, reliability analysis, graph theory, network problems, to mention but a few. For US financial markets, [6] developed a stock model with jumps. This model was later applied in solving a US option pricing problem in the existence of uncertain jumps by [44]. Recently, [45] proposed a US optimal control model with a jump and applied it to portfolio game symmetry. [46] tackled a multi-objective optimization problem in uncertain random environments. [47] applied US systems with Markovian switching in solving a portfolio selection problem. Interestingly, until now, no study has designed an IR model for US financial markets.

In this study, we suggest that the short rates of interest are driven by USDEs in the US financial markets. In this framework, the presumption is that the IR is driven by two sources of uncertainty. Employing an uncertain differential equation, we design a US IR model and then implement that model to price a zero-coupon bond. For effective and efficiency purposes, numerical examples concerning zero-coupon bond pricing are presented.

The rest of the article is arranged as follows: Section 2 covers the preliminaries, and Section 3 examines a US short-rate model. Numerical examples are outlined in Section 4. Conclusions are given in Section 5.

Preliminaries

This section outlines crucial definitions concerning uncertainty theory, probability theory and chance theory. We presume a complete filtered uncertainty probability space $(\Gamma \times \Omega,\ \mathcal{L} \times \mathcal{F},\ \mathcal{M} \times P)$ characterised by a filtration $(\mathcal{L} \times \mathcal{F}_t)_{t\in[0,T]}$, created by a standard a one-dimensional Brownian motion, $\{W_t\}_{t\in[0,T]}$, and a one-dimensional Liu process, $\{C_T\}_{\in[0,T]}$. Basically, $\Gamma \times \Omega$ represents the universal set, $\mathcal{L} \times \mathcal{F}$ is a product σ-algebra, $\mathcal{M} \times P$ signifies a product measure and $(\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times P)$ denotes an uncertainty probability space.

Definition 1 Presume Ω denotes a non-empty set and $\mathcal{F}$ represents a σ-algebra over Ω. Every component A in $\mathcal{F}$ is an event. A probability measure refers to a set function $P : \mathcal{F} \to [0,1]$ which fulfills the below-stated axioms:

- **Normality:** $P\{\Omega\} = 1$ for the universal set Ω
- **Non-negativity:** $P\{A\} \geq 0$ for every event A;
- **Additivity:** For each countable series of mutually disjoint events

$$A_1, A_2, A_3, \ldots,\ P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\}.$$

Definition 2 A random variable refers to a function ε from a probability space $(\Omega, \mathcal{F}, P)$ to a set of real numbers in a manner that for each Borel set B of real numbers, $\{\varepsilon \in B\}$ is regarded as an event.

Definition 3 Presume $(\Omega, \mathcal{F}, P)$ denotes a probability space, and T is an index set. A stochastic process refers to a measurable function $X_T(\omega)$ from $T \times (\Omega, \mathcal{F}, P)$ to the set of real numbers in a manner that at any time *t*, for every Borel set of real numbers, $\{X_t \in B\}$ is an event. Basically, a stochastic process refers to a series of random variables indexed by space or time.

Definition 4 A stochastic process W_t is regarded as a standard Brownian motion if

- $W_0 = 0$, and almost all sample paths are continuous,
- W_t *is* associated with independent and stationary increments,
- each increment $W_{s+t} - W_s$ is a normal random variable with variance *t* and an expected value 0.

Definition 5 [4] Presumed that Γ denotes a non-empty set, and $\mathcal{L}$ represents a σ-algebra over Γ. Each component $\wedge$ in $\mathcal{L}$ is regarded as an event. Basically, an uncertain measure refers to a set function $\mathcal{M} : \mathcal{L} \to [0,1]$ which fulfills the below-stated axioms:

- **Normality:** $\mathcal{M}\{\Gamma\} = 1$;
- **Monotonicity:** $\mathcal{M}\{\wedge_1\} \leq \mathcal{M}\{\wedge_2\}$ if $\wedge_1 \subset \wedge_2$;
- **Duality:** $\mathcal{M}\{\wedge_1\} + \mathcal{M}\{\wedge^c\} = 1$ for each $\wedge \in \mathcal{L}$;
- **Sub-additivity:** For each countable event series $\{\wedge_1, \wedge_2, \ldots\}$, $\mathcal{M}\left\{\bigcup_i \wedge_i\right\} \leq \sum_i \mathcal{M}\{\wedge_i\}$.

Definition 6 [4] An uncertain variable refers to a measurable function ε from an uncertainty space $(\Gamma,\ \mathcal{L}, \mathcal{M})$ to the set of real numbers in a manner that for every Borel set B, $\{\varepsilon \in B\}$ is regarded as an event.

Definition 7 [37] Presume that T is regarded as an index set, and $(\Gamma,\ \mathcal{L}, \mathcal{M})$ denotes an uncertainty space. Conceptually, an uncertain process refers to a measurable function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers in a manner that at any time t, for each Borel set B, $\{X_t \in B\}$ is regarded as an event.

Definition 8 [5] An uncertain process C_t is a Liu process if

- $C_0 = 0$, and almost all sample paths are Lipschitz continuous,
- C_t is associated with independent and stationary increments,

- each increment $C_{s+t} - C_s$ is regarded as a normal uncertain variable characterised by variance t^2 and expected value 0, whose uncertainty distribution is described by

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, x \in \mathcal{R}.$$

Definition 9 [37] Assume that $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, P)$ denotes a chance space, and $\Theta \in \mathcal{L} \times \mathcal{F}$ represents an uncertain random event. Hence, a chance measure $Ch\{\Theta\}$ is given by

$$Ch\{\Theta\} = \int_0^1 P\{\omega \int \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq r\} dr.$$

A chance measure fulfills the following axioms:

- **Normality:** (Liu, 2013) $Ch(\Gamma \times \Omega) = 1,\ \mathrm{Ch}\{\varnothing\} = 0;$
- **Monotonicity:** (Liu, 2013) $Ch\{\Theta_1\} \leq Ch\{\Theta_2\}$ for each event $\Theta_1 \leq \Theta_2$;
- **Self-duality:** (Liu, 2013) $Ch\{\Theta\} + Ch\{\Theta^c\} = 1$ for each event Θ;
- **Sub-additivity:** (Hou, 2014) For each countable series of events $\Theta_1, \Theta_2 \ldots, Ch\left\{\bigcup_i^\infty \Theta_i\right\} = \sum_i^\infty Ch\{\Theta_i\};$

- **Null-additivity:** (Hou, 2014) Assume that $\Theta_1, \Theta_2 \ldots,$ denotes a series of events with $Ch\{\Theta_i\} \to 0$ as $i \to \infty$. So, for each event

$$\lim_{i\to\infty} Ch\{\Theta \cup \Theta_i\} = \lim_{i\to\infty} Ch\left\{\frac{\Theta}{\Theta_i}\right\} = Ch\{\Theta\}.$$

This implies that $Ch\{\Theta_1 \cup \Theta_2\} = Ch\{\Theta_1\} + Ch\{\Theta_2\}$ if either $Ch\{\Theta_1\} = 0$ or $Ch\{\Theta_2\} = 0;$

- **Axiom 6: Asymptotic** (Hou 2014) For each series of events $\Theta_1, \Theta_2, \ldots,$

$$\lim_{i\to\infty} Ch\ \{\Theta_i\} > 0, if\ \Theta_i \uparrow \Gamma \times \Omega,\ \lim_{i\to\infty} Ch\ \{\Theta_i\} < 1, if\ \Theta_i \downarrow \varnothing.$$

Definition 10 [37] An uncertain random variable refers to a measurable function ξ from a chance space $\{\Gamma, \mathcal{L}, \mathcal{M}\} \times \{\Omega, \mathcal{F}, P\}$ to the real numbers' set in a manner that for any Borel set B of real numbers, the set $\{\xi \in B\} = \{(\gamma, \omega) \mid \xi(\gamma, \omega) \in B\}$ is regarded as an uncertain random event in $\mathcal{L} \times \mathcal{F}$.

Definition 11 [42] (*i*) An uncertain random variable refers to a measurable function ξ from an uncertainty probability space $(\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times P)$ to the real numbers' set in a manner that for each Borel set B of real numbers, the set

$$\{\xi \in B\} = \{(\gamma, \omega) \in \Gamma \times \Omega \mid \xi(\gamma, \omega) \in B\} \in \mathcal{L} \times \mathcal{F}$$

is regarded as an event.

(ii) The expected value of an uncertain random variable ξ is described by

$$E[\xi] = E_p[E_{\mathcal{M}}[\xi]]$$

given that the operations on the right-hand are described well. The operators

$$\mathrm{E}_{\mathcal{M}} \text{ and } \mathrm{E}_p$$

represent the expected values in the context of the probability space and uncertainty space, respectively.

Suppose b and a are constants, $E[aC_t + bW_t] = 0$, where W_t is a standard one-dimensional Brownian motion and C_t is a Liu process. Interestingly, the uncertain random variable definitions introduced by [37] and [42] are not similar. The definition propounded by [37] indicates that an uncertain random variable is generally a function from a probability space to a set of uncertain variables [42].

Definition 12 [41] Assume that T is a completely ordered set and $(\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times P)$ refers to a chance space. Conceptually, an uncertain random process refers to a measurable function $X_t(\gamma, \omega)$ from $T \times \{\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times P\}$ to the set of real numbers in a manner that the set

$$\{X_t \in B\} = \{(\gamma,\omega) \in \Gamma \times \Omega \mid X_t(\gamma,\omega) \in B\}$$

is regarded as an uncertain random event in $\mathcal{L} \times \mathcal{F}$ for every Borel set B of real numbers at any time $t \in T$. Basically, an uncertain random process refers to a series of uncertain random variables indexed by time or space, $t \in [0, \infty)$ and is described on a chance space $\{\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times P\}$.

Uncertain stochastic short-rate model

This study assumes that the IR can be explained by an USDE. Hence, a US IR model to explain the dynamics of the IR is examined under this framework. An US IR model in US markets can be explained by a USDE

$$dr_t = e(t,r_t)dt + \sigma_1(t,r_t)dW_t + \sigma_2(t,r_t)dC_t,$$

where the short IR at time *t* is given by *r*, and the drift *e*, stochastic diffusion σ_1 and uncertain diffusion σ_2 are presumed to be functions of *t* and *r*. This model is an expansion of the model suggested by [6] in 1973. It is a sole factor short-term IR model that represents the evolution of the IR r_t in the presence of epistemic and aleatory uncertainty. The model lacks mean reversion, the IR is an uncertain random variable, and the model has two sources of uncertainty, that is, epistemic and aleatory uncertainty. Aleatory uncertainty is premised on the random experiment outcomes' unpredictability, while epistemic uncertainty is powered by the deficiency of adequate or precise knowledge about facts. In the following section, the model is going to be used in zero-coupon bond pricing.

Pricing a zero-coupon bond using an uncertain stochastic short rate model

[48] indicated that if $P(t,T)$ is the price of the zero-coupon bond associated with a maturity date *T*, then if today is time *t*, the maturity time can be defined as $\tau = T - t$. Also, [31] propounded that the τ-period spot IR at time *t* is described by $s(t,T)$. In addition, [48] indicated that the τ-period spot rate of interest at time *t* satisfies the equations

$$P(t,T) = \exp(-s(t,T)(T-t)),$$

and

$$s(t,T) = -\frac{\ln P(t,T)}{(T-t)}. \tag{1}$$

Further, [48] propounded that, suppose $\mathrm{f}(t,T)$ denotes a forward IR deduced from the zero-coupon bond, then

$$P(t,T) = \exp\left(-\int_t^T \mathrm{f}(t,T)\,\mathrm{d}s\right),$$

and

$$\mathrm{f}(t,T) = -\frac{\frac{\partial P(t,T)}{\partial T}}{P(t,T)}. \tag{2}$$

If the short IR is presumed to follow a USDE, the zero-coupon bond price $P(t,T)$ is the anticipated value of one dollar discounted by the probable short rate process's paths. [48] indicated that the local equilibrium hypothesis is that the expected immediate return is given by the short rate of the form

$$\frac{E[P(t+\Delta t,T)]}{P(t,T)} = \exp(r_t \Delta t), \text{ as } \Delta t \to 0.$$

From the local equilibrium hypothesis, the zero-coupon bond price $P(t,T)$ is described by

$$P(t,T) = E\left[\exp\left(-\int_t^T r_v\,\mathrm{d}v\right)\right]. \tag{3}$$

The savings account is represented by

$$\beta(t) = \exp\left(\int_0^t r(s)\,ds\right).$$

Theorem 1: Suppose a short rate is described by an USDE of the form

$$dr_t = e_t dt + \sigma_1 dW_t + \sigma_2 dC_t, \tag{4}$$

where e_t is a function of t and σ_1 *and* σ_2 denote the constants, then the price of the zero-coupon bond associated with a maturity date T is given by

$$P(0,T) = \frac{1}{2}\sqrt{3}\sigma_2 T^2 \exp\left(-r_0 T \int_0^t \int_0^s e_t dt ds - \sigma_1 T \rho\right) \csc\left(\frac{1}{2}\sqrt{3}\sigma_1 T^2\right). \tag{5}$$

Proof: Equation 4 is an extension of Merton's model, whose solution, r_t, is of the form

$$r_t = r_0 + \int_0^t e_s ds + \sigma_1 W_t + \sigma_2 C_t.$$

The expected value of r_t is

$$E[r_t] = E\left[r_0 + \int_0^t e_s ds + \sigma_1 W_t + \sigma_2 C_t\right] = r_0 + \int_0^t e_s ds$$

and its variance is described by

$$Var[r_t] = Var\left[r_0 + \int_0^t e_s ds + \sigma_1 W_t + \sigma_2 C_t\right] = \sigma_1^2 t + \sigma_2^2 t^2.$$

From equation 3, for $t = 0$,

$$P(0,T) = E\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right)\right] = \exp\left[\int_0^\infty \mathcal{M}[\int_0^T r_s \mathrm{d}s \le -x]\right]\mathrm{d}x =$$

$$= \exp[\int_0^\infty \mathcal{M}[r_0 T + \int_0^t \int_0^s e_t dt ds + \sigma_1 T W_t + \sigma_2 T C_t \le -x\ ]\mathrm{d}x] =$$

$$= \exp[\int_0^\infty [\mathcal{M}[C_t \le \frac{-x - r_0 T - \int_0^T \int_0^s e_t dt ds - \sigma_1 T W_t}{\sigma_2 T}]]dx] =$$

$$= \exp\left[\int_0^\infty \left[\left(1 + \exp\left(-\frac{\pi\left(-x - r_0 T - \int_0^T \int_0^s e_t dt ds - \sigma_1 T W_t\right)}{\sqrt{3}\sigma_2 T^2}\right)\right)^{-1}\right] dx\right].$$

Replacing W_t with its realizations ρ, we have

$$P(0,T) = \left[\int_0^\infty \left[\left(1 + \exp\left(-\frac{\pi\left(-x - r_0 T - \int_0^T \int_0^s e_t dt ds - \sigma_1 T \rho\right)}{\sqrt{3}\sigma_2 T^2}\right)\right)^{-1}\right] dx\right] =$$

$$= \frac{1}{2}\sqrt{3}\sigma_2 T^2 \exp\left(-r_0 T - \int_0^T \int_0^s e_t dt ds - \sigma_1 T \rho\right) \csc\left(\frac{1}{2}\sqrt{3}\sigma_2 T^2\right) \tag{6}$$

Thus, the proof is concluded since equation 6 equals equation 5 which is the price of the zero-coupon bond. Replacing e_t in equation 6 with a constant e gives

$$P(0,T)=\frac{1}{2}\sqrt{3}\sigma_2 T^2\exp\left(-r_0T-\frac{1}{2}eT^2-\sigma_1T\rho\right)\csc\left(\frac{1}{2}\sqrt{3}\sigma_2T^2\right). \tag{7}$$

Applying equations 1 and 2 to equation 7, the forward IR is given by

$$S(0,T)=-\frac{\ln P(0,T)}{T}=$$
$$=-\frac{\ln\left(\frac{1}{2}\sqrt{3}\sigma_2T^2\right)}{T}+\frac{1}{2}eT+r_0+\sigma_1\rho-\frac{\ln\left(\csc\left(\frac{1}{2}\sqrt{3}\sigma_2T^2\right)\right)}{T}, \tag{8}$$

and the spot IR is described by

$$\mathrm{f}(0,T)=-\frac{\frac{\partial P(0,T)}{\partial T}}{P(0,T)}=$$
$$=-\frac{2}{T}+eT+r_0+\sigma_1\rho+\sqrt{3}\sigma_2T\ \csc\left(\frac{1}{2}\sqrt{3}\sigma_2T^2\right). \tag{9}$$

Uncertain stochastic mean reverting short rate model

In this section, an US mean reverting short rate model is proposed. Consider a linear USDE.

Theorem 2: Let $a_{1t}, a_{2t}, b_{1t}, b_{2t}, c_{1t}, c_{2t}$ be integrable uncertain random processes. The linear USDE

$$\mathrm{d}X_t=(a_{1t}X_t+a_{2t})\mathrm{d}t+(b_{1t}X_t+b_{2t})\mathrm{d}W_t+(c_{1t}X_t+c_{2t})\mathrm{d}C_t \tag{10}$$

has solution

$$X_t=U_t\left(X_0+\int_0^t\frac{a_{2s}}{U_s}\mathrm{d}s+\int_0^t\frac{b_{2s}}{U_s}\mathrm{d}W_s+\int_0^t\frac{c_2s}{U_s}\mathrm{d}C_s\right) \tag{11}$$

where

$$U_t=\exp\left(\int_0^t a_{1s}\mathrm{d}s+\int_0^t b_{1s}\mathrm{d}W_s+\int_0^t c_{1s}\mathrm{d}C_s\right).$$

Proof: Let V_t and U_t be two US processes such that

$$dU_t=a_{1t}U_tdt+b_{1t}U_tdW_t+c_{1t}U_tdC_t,$$

$$\mathrm{d}V_t=\frac{a_{2t}}{U_t}\mathrm{d}t+\frac{b_{2t}}{U_t}\mathrm{d}W_t+\frac{c_{2t}}{U_t}\mathrm{d}C_t.$$

From integration by parts,

$$\mathrm{d}(U_tV_t)=V_t\mathrm{d}U_t+U_t\mathrm{d}V_t=(a_{1t}U_tV_t+a_{2t})\mathrm{d}t\ +(b_{1t}U_tV_t+b_{2t})\mathrm{d}W_t+(c_{1t}U_tV_t+c_{2t})\mathrm{d}C_t.$$

An uncertain random process in equation 11 given by $X_t=U_tV_t$ is a solution to equation 10 where

$$U_t=U_o\exp\left(\int_0^t a_{1s}\mathrm{d}s+\int_0^t b_{1s}dW_s+\int_0^t c_{1s}\mathrm{d}C_s\right)$$

and

$$V_t=V_0+\int_0^t\frac{a_{2s}}{U_s}ds+\int_0^t\frac{b_{2s}}{U_s}dW_s+\int_0^t\frac{c_{2s}}{U_s}dC_s.$$

If

$$V_0=X_0 \text{ and } U_0=1,$$

theorem 2's solution is obtained; thus, the proof is concluded.

An US mean reverting IR model in US markets can be described by an USDE

$$dr_t = (m - er_t)dt + \sigma_1 W_t + \sigma_2 C_t,$$

where m, e, σ_1 and σ_2 are constants. This model is an expansion of the model suggested by [7] in 1977. It is a sole factor short term IR model that represents the evolution of the IR r_t in the presence of aleatory and epistemic uncertainty. The model incorporates mean reversion to the dynamics of the IR process, the IR is presumed to be an uncertain random variable and the model has two sources of uncertainty, as in the previous sections. A zero-coupon bond is priced under this framework in the following section.

Pricing a zero-coupon bond using a mean reverting uncertain stochastic short rate model

Here, a zero-coupon bond pricing model is examined in the framework of a mean reverting US short rate model.

Theorem 3: Suppose the short rate is explained by an USDE

$$dr_t = (m - er_t)dt + \sigma_1 W_t + \sigma_2 C_t, \tag{12}$$

where m, e, σ_1 and σ_2 represent constants, the price of the zero-coupon bond associated with a maturity date T is given by

$$P(0,T) = \beta\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp(-eT)\right) \times \exp\left(-\frac{mT}{2e} - \frac{1}{e}\left(r_0 - \frac{m}{e}\right)\left(1 - \exp(-eT)\right)\right) \times$$

$$\times \csc\left(\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp(-aT)\right)\right),$$

where

$$\beta = \exp\left(-\left[\int_0^T (\exp(-es)\, E(\int_0^T \exp(es) dW_s)) ds\right]\right).$$

Proof: Equation 12 is the Vasicek model's extension. The process r_t includes mean reversion. Applying theorem 2 to solve equation 12, we have

$$a_{1t} = -e, a_{2t} = m, b_{1t} = 0,\ b_{2t} = \sigma_1, c_{1t} = 0,\ and\ c_{2t} = \sigma_2,$$

which means $U_t = \exp(-et)$. This indicates

$$r_t = U_t\left(r_0 + \int_0^t \frac{m}{U_s} ds + \int_0^t \frac{\sigma_1}{U_s} dW_s + \int_0^t \frac{\sigma_2}{U_s} dC_s\right) =$$

$$= \exp(-et)\left(r_0 + \int_0^t m \exp(es) ds + \int_0^t \sigma_1 \exp(es) dW_s + \int_0^t \sigma_2 \exp(es) dC_s\right).$$

Alternatively,

$$d(\exp(et) r_t) = \exp(et) r_t dt + \exp(et)(mdt - r_t dt + \sigma_1 dW_t + \sigma_2 dC_t) =$$

$$= \exp(-et) mdt + \exp(-et)\sigma_1 dW_t + \exp(-et)\sigma_2 dC_t.$$

That is

$$\exp(et) r_t = r_0 + m\int_0^t \exp(es) ds + \sigma_1 \int_0^t \exp(es) dW_s + \sigma_2 \int_0^t \exp(es) dC_s \cdot$$

This means

$$r_t = r_0\exp(-et) + m\int_0^t \exp(es - et)ds + \sigma_1\exp(-et)\int_0^t \exp(es)dW_s + \sigma_2\exp(-et)\int_0^t \exp(es)dC_s =$$
$$= r_0\exp(-et) + \frac{m}{e} - \frac{m}{e}\exp(-et) + \sigma_1\exp(-et)\int_0^t \exp(es)dW_s + \sigma_2\exp(-et)\int_0^t \exp(es)dC_s =$$
$$= \frac{m}{e} + \exp(-et)\left(r_0 - \frac{m}{e}\right) + \sigma_1\exp(-et)\int_0^t \exp(es)dW_s + \sigma_2\exp(-et)\int_0^t \exp(es)dC_s, \tag{13}$$

given that $e \neq 0$. The expected value of r_t in equation 13 above is desribed by

$$E[r_t] = \frac{m}{e} + \exp(-et)\left(r_o - \frac{m}{e}\right)$$

and the variance is described by

$$Var[r_t] = \frac{\sigma_1^2}{2}[1 - \exp(-2t)] + \frac{\sigma_2}{e} - \exp(-et)\frac{\sigma_2}{e}.$$

Applying the local equilibrium hypothesis, the following is deduced

$$P(0,T) = E\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right)\right] =$$
$$= \exp\left(-E\left[\int_0^T (\exp(-es)\int_0^T \exp(es)dW_s)ds\right] - E\left[\int_0^T (\frac{m}{e} + \exp(-es)\left(r_o - \frac{m}{e}\right) + \sigma_2\exp(-es)\int_0^T \exp(es)dC_s)ds\right]\right).$$

That is,

$$P(0,T) = \exp\left(-E\left[\int_0^T (\exp(-es)\int_0^T \exp(es)dW_s)ds\right]\right)\times$$
$$\times\exp\left(-E\left[\int_0^T (\frac{m}{e} + \exp(-es)\left(r_o - \frac{m}{e}\right) + \sigma_2\exp(-es)\int_0^T \exp(es)dC_s)ds\right]\right).$$

Which translates to

$$P(0,T) = \exp\left(-\left[\int_0^T (\exp(-es)E(\int_0^T \exp(es)\mathrm{d}W_s))\mathrm{d}s\right]\right)\times$$
$$\times\exp\left[\int_0^\infty \mathcal{M}[\int_0^T (\frac{m}{e} + \exp(-es)\left(r_o - \frac{m}{e}\right) + \sigma_2\exp(-es)\int_0^T \exp(es)dC_s)ds \leq -x]\right]dx.$$

Let

$$\beta = \exp\left(-\left[\int_0^T (\exp(-es)E(\int_0^T \exp(es)dW_s))ds\right]\right).$$

Thus,

$$P(0,T) = \beta\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp(-eT)\right)\times$$
$$\times\exp\left(-\frac{mT}{2e} - \frac{1}{e}\left(r_0 - \frac{m}{e}\right)(1 - \exp(-eT))\right)\times$$
$$\times\csc\left(\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp(-eT)\right)\right). \tag{14}$$

This concludes the proof since equation 14 is similar to the price of the zero-coupon bond in theorem 3. Klebaner (2005) indicated that if

$$\int_0^T E\left(X^2(t)\right)\mathrm{d}t < \infty,$$

then we have the zero-mean property which states that

$$E\left(\int_0^T X(t)\mathrm{d}W_t\right) = 0.$$

Since

$$\int_0^T E\left(\exp(2es\right)\mathrm{d}s < \infty, \quad E\left(\int_0^T \exp\left(es\right)\mathrm{d}W_s\right) = 0\,.$$

This means that by applying the zero-mean property, the zero-coupon bond price, instantaneous forward rate, and spot IR degenerate to the ones proposed by Chen (2016), that is

$$P\left(0,T\right) = \sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp\left(-eT\right)\right) \times \exp\left(-\frac{mT}{2e} - \frac{1}{e}\left(r_0 - \frac{m}{e}\right)\left(1 - \exp\left(-eT\right)\right)\right) \times$$
$$\times \csc\left(\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp\left(-eT\right)\right)\right),$$

$$S\left(0,T\right) = -\frac{\ln P\left(0,T\right)}{T} =$$

$$= -\frac{1}{T}\ln\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp\left(-eT\right)\right) + \frac{1}{T}\left(\frac{mT}{2e} + \frac{1}{e}\left(r_0 - \frac{m}{e}\right)\left(1 - \exp\left(-eT\right)\right)\right) -$$
$$-\frac{1}{T}\ln\left[\csc\left(\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp\left(-eT\right)\right)\right)\right], \tag{15}$$

and

$$\mathrm{f}\left(0,T\right) = -\frac{\partial P\left(0,T\right)}{\partial T} / P\left(0,T\right) =$$

$$= -\left(\frac{\sigma_2}{e^2} + e\frac{\sigma_2}{e^2}\exp\left(-eT\right)\right) / \left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp\left(-eT\right)\right) =$$

$$+\left(\frac{m}{2e} - \left(r_0 - \frac{m}{e}\right)\exp\left(-eT\right)\right) + \left(\sqrt{3}\frac{\sigma_2}{e^2} - \frac{\sigma_2}{e}\exp\left(-eT\right)\right) \times$$

$$\times \csc\left(\sqrt{3}\left(\frac{\sigma_2 T}{e} - \frac{\sigma_2}{e^2} + \frac{\sigma_2}{e^2}\exp\left(-eT\right)\right)\right).$$

Remark: In the long run, the IR converges to $\frac{m}{a}$ from r_0.

Numerical example

This section presents some numerical examples of the alpha path and zero-coupon bond pricing.

Example 1

Let the initial interest rate be $r_0 = 0.08$, the instantaneous drift be $e = 0.1$, $\sigma_1 = 0.15$ and $\sigma_2 = 0.2$, $T = 10$ years and $m = 0.002$. The aim is to compute the α – path of equation 4 and equation 10 using the Euler-Maruyama method.

The α – path of equation 4 is given by

$$dr_t^\alpha = \left(e + \sigma_1\rho + \sigma_2\Phi_t^{-1}\left(\alpha\right)\right)dt$$

and the alpha path for its solution is

$$r_t^\alpha = r_0 + et + \sigma_1\rho + \sigma_2\left(\frac{\sqrt{3}t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

In this case, ρ represents the realisations of a standard Wiener process, α is a measure of belief and $\Phi_t^{-1}\left(\alpha\right)$ is the inverse uncertainty distribution given in definition 8 and the other parameters are as previously defined. The alpha path represents the evolution of interest rates under a specific belief degree α.

Alpha ranges from 0 to 1. These paths provide insights into the range of possible outcomes based on different levels of beliefs. In other words, α – path captures the system's behavior at this specific belief degree. These paths are real-valued functions of time t, but are not necessarily one of the sample paths [43]. Alpha paths are derived from the inverse uncertainty distribution, which provides the value of the uncertain variable for each belief degree level.

Figure 1 represents the α – path graphs for an uncertain stochastic interest rate model in equation 4. Notice how the graphs diverge from each other for different alpha values. Also, note that r_t^α is a linear function of time. The α – path of equation 10 is given by

$$dr_t^\alpha = \left(\left(m - er_t^\alpha\right) + \sigma_1\rho + \sigma_2\Phi_t^{-1}(\alpha)\right)dt$$

and the α – path for the solution is represented by

$$r_t^\alpha = r_0 \exp(-et) + \left(1 - \exp(-et)\right)\left(\frac{m}{e} + \frac{\sigma_1\rho}{e} + \frac{\sigma_2}{e}\left(\frac{t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

Note that the α – path for the solution of the mean-reverting model is no longer a linear function as in the previous case. *Figure 2* displays the graphs for the α – path for equation 10.

The gradients of these graphs are reducing to zero as time progresses, indicating that they approach a minimum or a maximum value. In the next example, we price a typical zero-coupon bond using the proposed methods.

Example 2

Let $r_0 = 0.08$, the instantaneous drift $e = 0.1$, $\sigma_1 = 0.15$ and $\sigma_2 = 0.2$, $T = 4$ years and $m = 0.002$. The aim is to illustrate how the zero-coupon bond price evolves and the behavior of the instantaneous forward rate as time progresses.

After implementing the formulas in equations 7, 8, 14, and 15 into Python, the prices of the zero-coupon bond and the instantaneous forward rates over the years are obtained. The graph in *Fig. 3*, which is based on equation 7, shows the relationship between the zero-coupon bond price and the time to maturity under an uncertain stochastic short rate model. The zero-coupon bond price decreases as time to maturity increases, then it starts to increase again. The relationship is non-linear. The bond price is sensitive to changes in the short rate over time. Bond prices exhibit convexity, which can lead to non-monotonic behaviour. Also, the bond price decreases for shorter maturities due to the discounting effect and increases for long maturities due to mean reversion or declining rates.

The prices obtained from the model in *Fig. 3* are less than those on the model in *Fig. 4*. *Figure 4* shows the relationship between the zero-coupon bond price and the time to maturity under a mean reverting uncertain stochastic short rate model. It is based on equation 14.

Figure 4 produces a smoother curve than *Fig. 3.* Equation 14 is a variant of the Vasicek model with an additional Liu process. In contrast, equation 7 is an extension of Merton's model that considers the effects of the Liu process. The model in equation 14 captures the behaviour of rates to stabilise around a long term average while the model in equation 7 assumes that interest rates follow a random walk, and this model can end up producing unrealistic long term behavior, for instance, interest rates becoming extremely high or low without a limit. However, *Fig. 3 and 4* show that the prices from the two models do not differ with a greater magnitude.

Instantaneous forward rates in *Fig. 5 and 6* increase with time, then they start to decrease.

Figure 5 above shows that the instantaneous forward rates based on equation 8 are greater than those in *Fig. 6,* which is based on equation 15. Also note that the graphs based on the mean-reverting process are smoother than the ones based on Merton's model. The periods where the instantaneous forward rates are negative imply that zero-coupon bonds have negative returns. This can be caused by the central bank, which sets negative interest rates to stimulate growth in situations of deflation or stagnation.

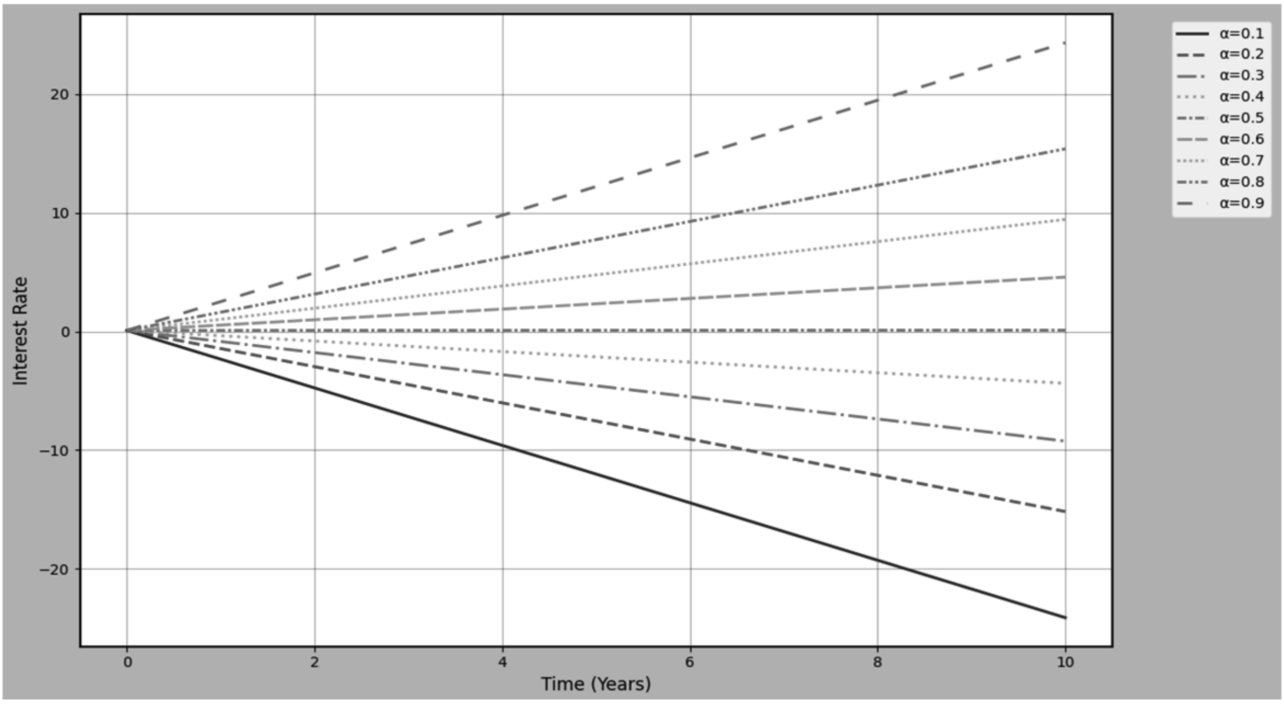

Fig. 1. **A Spectrum of** $\alpha-$ **path of** $dr_t = e_t dt + \sigma_1 dW_t + \sigma_2 dC_t$

Source: Developed by the authors.

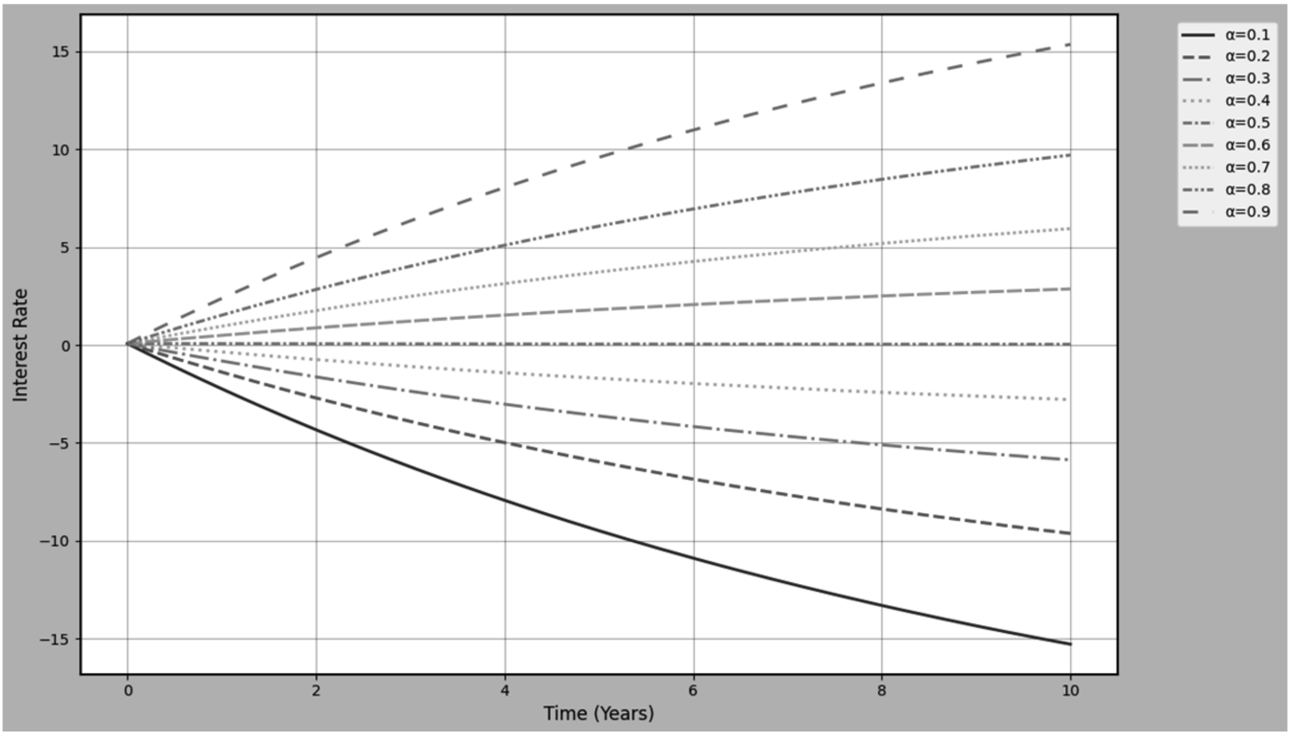

Fig. 2. **A Spectrum of** $\alpha-$ *paths* **of** $dr_t = (m - er_t)dt + \sigma_1 W_t + \sigma_2 C_t$

Source: Developed by the authors.

Also, this can be encountered in situations of financial stress. In addition, if markets foresee more negative interest rates, current forward rates may become negative. Apart from that, excess liquidity in the financial system can also contribute to negative instantaneous forward rates. In such situations, some investors can hold on to government bonds for their safety, while others may consider holding cash or other assets. Also, banks can struggle in making profits. These conditions are signals of weak economic growth or deflationary pressures.

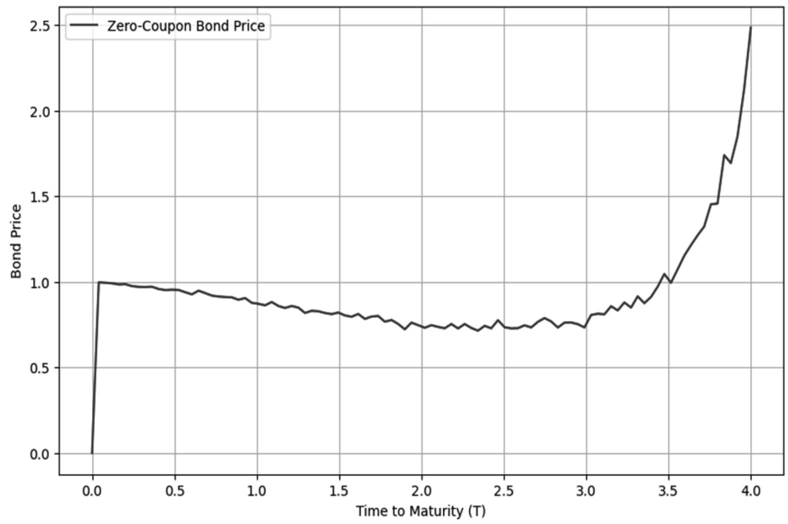

Fig. 3. **Price of zero-coupon bond against time to maturity for equation 7**

Source: Developed by the authors.

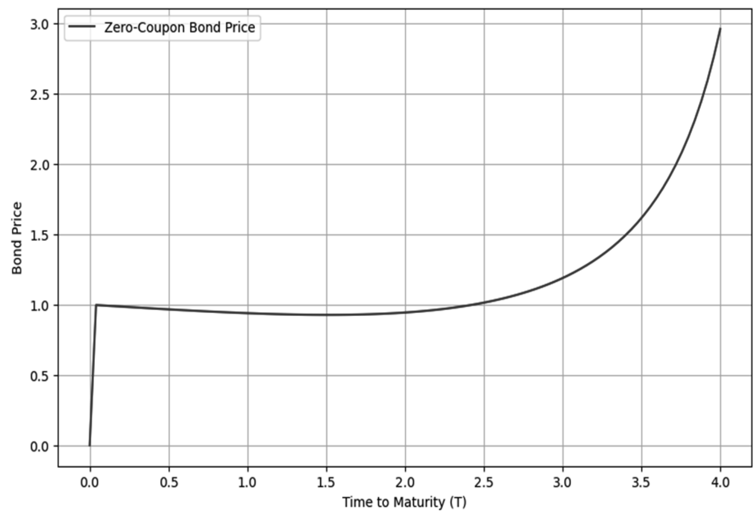

Fig. 4. **Price of zero-coupon bond against time to maturity for equation 14**

Source: Developed by the authors.

Conclusions

In this study, we suggested an interest rate model utilizing USDEs in the US financial markets. We also derived the model to compute the price of a zero-coupon bond for the interest rate model. To illustrate how to compute the price of the zero-coupon bond numerically, a practical example was presented. Modelling short-term interest rates and pricing zero-coupon bonds in the US environments represents a significant milestone in modelling financial markets under uncertainties. This approach is relevant in volatile or unpredictable market conditions. The US framework offers a flexible approach to pricing and risk management, allowing investors to combine epistemic uncertainty and exact probabilities into their models for zero-coupon bonds.

The proposed model can be applied to a wider range of emerging markets. The results obtained in this study can be extended beyond zero-coupon bond pricing to other IR derivatives. In addition, this approach can be integrated with machine learning to improve the prediction power. Also, this model can be applied in emerging markets, and policymakers and regulators can adopt this method in assessing the impact of interest rate shocks. However, the scope of this paper does not cover the use of real-world data. As part of our future work, an empirical validation will be carried out, and this will involve a rigorous comparative analysis of the developed model and established stochastic and uncertain models using real-world data in order to assess pricing accuracy and hedging effectiveness. In this process, we will estimate parameters and perform model calibration using real-world data.

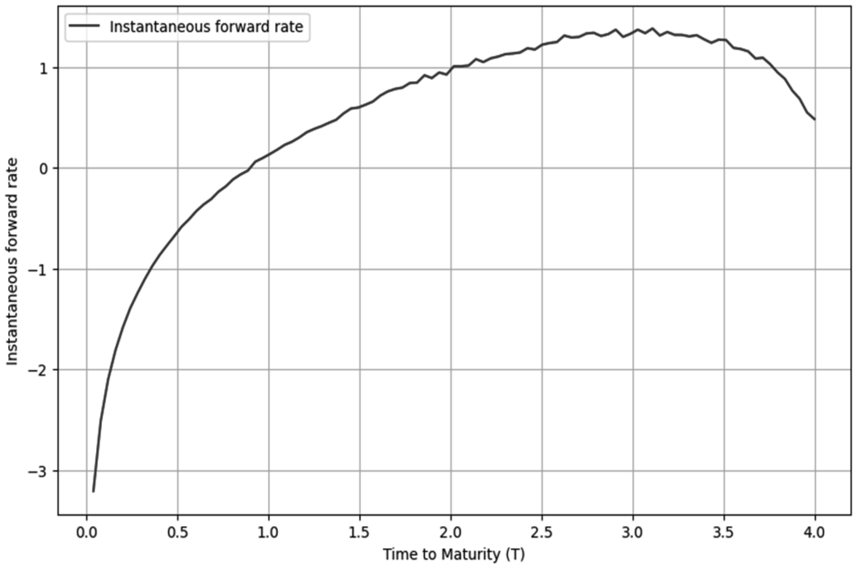

Fig. 5. **Instantaneous forward rate against time to maturity for equation 8**

Source: Developed by the authors.

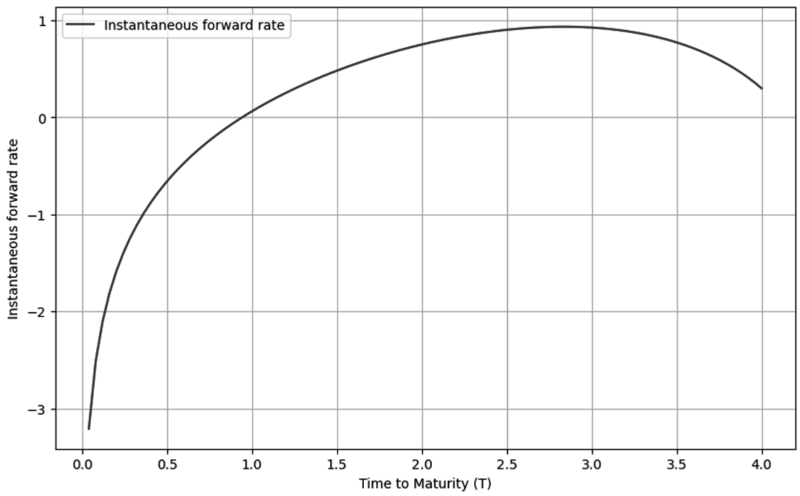

Fig. 6. **Instantaneous forward rate against time to maturity for equation 15**

Source: Developed by the authors.

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